MIURA TRANSFORMATIONS AND SYMMETRY GROUPS OF DIFFERENTIAL EQUATIONS

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Abstract

The main aim of this work is to develop a generalization of the Miura transformation based on symmetry groups of differential equations. Some applications of the resulting transformations are presented.

The central idea is a construction, called an HC-projection, which will generalize the well known Hopf-Cole transformation. With every differential equation there will be associated certain geometric objects which can be represented by new differential equations. These new differential equations inherit symmetries from the original differential equation. The reductions of these new differential equations using symmetry groups inherited from the original equation yield differential equations related to the original one by HC-projections.

HC-projections lead to special types of Wahlquist-Estabrook prolongation and various other geometric structures. Two differential equations will be said to be related by an M-projection if they are each related to a common differential equation by HC-projections with one generating symmetry group containing the other as a subgroup. These M-projections will generalize the Miura transformation. They lead to a wider class of Wahlquist-Estabrook prolongations than do HC-projections.

Construction of M-projections involves special Wahlquist-Estabrook prolongations of one of the differential equations related by the M-projection. These prolongations are characterized by their symmetry groups. A generalization of symmetries acting on nonlocal variables aids the construction of suitable prolongations, as do certain recursion operators. The construction of auto-Bäcklund transformations is significantly enhanced.

Contents

List of Tables

List of Figures

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Chapter 1

Introduction

The aim of the research presented here is to study analogues of the Miura transformation [65]. This transformation is used to map solutions between the Korteweg-de Vries (KdV) and modified Korteweg-de Vries (mKdV) equations, which are nonlinear evolution equations able to be solved by the inverse scattering method [1]. There is currently little agreement as to exactly what constitutes such "integrable" equations and various tests have been proposed for determining integrability [28]. Integrable equations seem to share many properties of the KdV equation, including possession of infinitely many symmetries [71], infinitely many conserved quantities [66] and infinite-dimensional prolongation algebras [92]. Many of these equations can be written in Hamiltonian form in two distinct ways [64] and admit recursion operators [71]. They admit auto-Bäcklund transformations [91] and reducing the equations to ordinary differential equations using symmetry groups yields reduced equations of Painlevé type [2]. Furthermore, many possess the Painlevé property as it has been formulated for partial differential equations [95]. The difficulty in making a precise definition of integrable equations is due to the fact that, although these properties are common to many equations which one would regard as being integrable, there are other equations, also integrable, which do not possess particular properties. For example, Burgers' equation has only one nontrivial conservation law (Example 5.34 of [72]), but is related to the heat equation and is thus integrable. Likewise, the Harry Dym equation can be solved by inverse scattering [89], but does not possess the Painlevé property for partial differential equations [93].

Motivation for studying the Miura transformation is provided by the observation that appropriate generalizations of the Miura transformation play an important role in most, if not all, of the properties listed above. For example, the Miura transformation can be used to construct an infinite family of conservation laws for the KdV equation [66]. More generally, analogues of the Miura transformation are used in [58] to construct differential equations which admit a bi-Hamiltonian formulation. Equations with this property possess recursion operators [64] which then yield infinite families of conservation laws and symmetries. Miura transformations also play an important part in many auto-Bäcklund transformations [10] and can lead, usually after linearizing a Riccati equation, to the linear equations involved in the inverse scattering process [34].

Miura transformations are often studied solely in terms of these special integrable differential equations. Examples include [58], mentioned above, as well as the construction of generalized Miura transformations relating solutions of analogues of the KdV and mKdV equations [20]. The starting point of [20] is any loop algebra over a finite-dimensional simple Lie algebra, and the equations derived admit a bi-Hamiltonian formulation. There is a need, however, to investigate the Miura transformation and its generalizations in a setting which encompasses all differential equations, and not just those equations one might regard as being integrable. That is, the Miura transformation must be studied only using structures which are common to all differential equations.

This thesis begins such an investigation using the geometric approach to differential equations. In particular, it exploits the symmetry structure of such equations. A symmetry group of a differential equation is a local group of transformations which maps solutions of that equation into other solutions. Infinitesimal generators of symmetry groups are determined by systems of linear differential equations. The advent of algebraic computing has enabled the symmetry algebras of many equations to be completely determined.

There are thus two components to this study. The first involves the wonderfully rich structures associated with integrable differential equations. Of course, this aspect is handicapped by its restriction to this class of equations. The lack of a definitive and constructive classification of these equations is another limiting factor. The second component is based on the computationally attractive notion of symmetry groups of differential equations. These structures can be associated with any differential equation but are rather bland by comparison with the structures arising from integrable systems. The research presented here attempts to blend these two approaches together. It does this by extending the notion of Miura transformation, such an important tool when investigating integrable equations, to a much wider class of differential equations. A simple, symmetry group-theoretic interpretation of these Miura transformations is provided. The author hopes that this approach to the Miura transformation will aid further study of integrable systems. It has already enriched the algebraic structures associated with some recursion operators (Sections 6.2 and 6.3) and enhanced the construction of auto-Bäcklund transformations (Section 5.5).

Notation and terminology are established in Chapter 2, which describes the geometric structures used subsequently. The presentation of jet spaces, differential equations and symmetry groups follows that of Olver [72] as closely as possible, while the approach to the prolongation method of Wahlquist and Estabrook [92] borrows heavily from the work of later authors [15], [76]. A reasonably self-contained description of the prolongation technique is included, as the method has been adapted to suit the aims of the present study.

Foundations for this investigation of the Miura transformation are laid in Chapter 3. The Hopf-Cole transformation relates solutions of Burgers' equation and the heat equation in much the same way as the Miura transformation relates solutions of the KdV and mKdV equations [13], [44]. Section 3.1 discusses the history of this transformation, while the study of its underlying geometry is pursued in Section 3.2. A certain geometric problem, called an extended problem, is uncovered. It amounts to constructing surfaces which foliate into leaves representing solutions to the heat equation. Extended problems can be defined for any system of differential equations and can themselves be represented by differential equations — called extended equations here. Each solution to an extended equation yields a multi-parameter family of solutions to the parent differential equation. Section 3.3 shows that all extended problems, and hence extended equations, inherit symmetry groups from their parent differential equations. Furthermore, reducing an extended equation associated with the heat equation using one such symmetry group yields Burgers' equation. Generalizations of the Hopf-Cole transformation are obtained from every symmetry group of a differential equation (meeting some technical requirements) by reducing some appropriate extended equation. Solutions can be mapped between the new equation and the original one in a manner analogous to that for the Hopf-Cole transformation. Transformations constructed in this way are called HC-projections. This demonstrates the useful role that geometric considerations can play in studying differential equations. An analytic problem, the construction of generalized Hopf-Cole transformations, has been attacked using the geometric concept of symmetry groups. The classification of HC-projections is a transparently group-theoretic process. As well as yielding generalized Hopf-Cole transformations, extended problems may prove useful in other areas. Section 3.4 suggests how extended problems can assist in investigating some solution-generating techniques — in particular, the use of side conditions [73] and the construction of partially-invariant solutions [75]. Finally, Section 3.5 considers the construction of auto-Bäcklund transformations using HC-projections. The Bäcklund transformations obtained can be derived solely from symmetry group properties.

The next step towards a generalization of the Miura transformation is taken in Chapter 4. Subject to one technical requirement, each HC-projection leads to a special type of Wahlquist-Estabrook prolongation. This observation, pursued in Section 4.1, generalizes the well known result that the heat equation can be recovered from a prolongation of Burgers' equation. The Wahlquist-Estabrook prolongations which occur in this way are characterized by their symmetry properties. Another structure which can be associated with HC-projections is described in Section 4.2. Solutions to equations arising as HC-projections of some differential equation lead to flat connections on principal fibre bundles. The symmetry groups which generate these HC-projections appear as the structure groups of these fibre bundles. The special Wahlquist-Estabrook prolongations of the first section are used in Section 4.3 to decompose HC-projections into sequences of simpler transformations. If the symmetry group G of a differential equation contains a normal subgroup H then the HC-projection induced by G can be decomposed into an HC-projection, induced by H, onto some intermediate equation, followed by another HC-projection relating the intermediate equation and the one occurring as the G-induced projection. The second HC-projection is generated by a symmetry group $K \cong G/H$ of the intermediate equation. Section 4.4 reviews the Miura transformation and derives it using HC-projections of the singularity manifold equation of the KdV equation [93]. It is shown that the KdV and mKdV equations are related to the singularity manifold equation by HC-projections generated by three- and two-dimensional symmetry groups respectively. The fact that the two-dimensional group is a subgroup, but not a normal subgroup, of the other one motivates the generalization of the Miura transformation which is used here. Two differential equations are said to be related by an M-projection if they are related to a common equation by HC-projections induced by a symmetry group, and one of its subgroups, of the latter equation. Solutions can be mapped between the two equations in a manner remarkably similar to that arising from HC-projections. As with HC-projections, M-projections lead to Wahlquist-Estabrook prolongations and connections on fibre bundles. However, a larger class of Wahlquist-Estabrook prolongations arises from M-projections and the fibre bundles possessing the connections now need not be principal bundles. The final section in Chapter 4 constructs all evolution equations related to the singularity manifold equation of the KdV equation by HC-projections. Such a construction amounts to a straightforward group classification problem and suggests that the special Wahlquist-Estabrook prolongations associated with HC-projections provide a structure suitable for developing a generalization of Galois theory to differential equations.

Chapter 5 attacks the problem of constructing M-projections given only the analogues of the mKdV equation. This task is made difficult by the need to first identify the equation, analogous to the singularity manifold equation of the KdV equation, which admits the given differential equation and the analogue of the KdV equation as HC-projections. Section 5.1 restates this problem in terms of Wahlquist-Estabrook prolongations of the given differential equation and their symmetry structures. The notion of symmetry is generalized in Section 5.2 in order to ease the search for prolongations meeting the requirements of the first section. Such prolongations can often be found more easily when the differential equation under investigation admits a recursion operator. Section 5.3 motivates the need for, and then presents, an alternative definition of recursion operator adapted for the purposes of the preceding section. This new notion of recursion operator is especially useful as it facilitates the use of the inverses of some famous recursion operators of differential equations. The construction of M-projections is summarized in Section 5.4 which also provides an efficient computational procedure for deriving the projected equation. The construction of M-projections will sometimes break down, with the prolongation of the given differential equation meeting only some of the symmetry requirements of Section 5.1. Section 5.5 examines this situation further and demonstrates how one can obtain new equations related to the original one by Bäcklund transformations. In special cases the new equation coincides with the original one, so that one actually has an auto-Bäcklund transformation of the system being studied. This promising new technique involves infinitesimal methods, only, and significantly enhances existing approaches to the problem of discovering auto-Bäcklund transformations for differential equations. A new auto-Bäcklund transformation for the Harry Dym equation is found using this method.

Chapters 3 to 5 have concentrated primarily on developing the structures which allow symmetry groups to be used to construct generalized Hopf-Cole and Miura transformations for any differential equation. Chapter 6 focuses on the other main aspect of this study — the rich algebraic structures associated with integrable equations. The results of earlier chapters are used to add one other algebraic structure associated with the KdV equation to the list presented at the beginning of this introduction. Section 6.1 constructs a sequence of prolongations of the KdV equation. The limiting prolongation can also be derived from a representation of the prolongation algebra usually associated with the KdV equation [21]. Section 6.2 inverts the recursion operator $\mathcal{R}$ of the KdV equation, applies the result to the zero symmetry characteristic and uses the technique of Section 5.2 to construct infinitely many symmetry generators of the prolonged system. These symmetries span the loop algebra over $\mathfrak{sl}(2,\mathbb{R})$. Thus, the action of $\mathcal{R}^{-1}$ on zero generates $\mathfrak{sl}(2,\mathbb{R})\otimes\mathbb{R}[\lambda,\lambda^{-1}]$ — an exciting result, when one considers the role that loop algebras play in many integrable equations [20], [25], [26], [42], [97]. Following suitable coordinate changes, Section 6.3 extends this result to the mKdV equation and two other equations related to the KdV equation.

Appendix A compares the approach to the Hopf-Cole transformation adopted here with the work of Sokolov, Svinolupov and Wolf [87], which appeared after much of this thesis had been written [36]. Material which could not conveniently be included in the main text is presented in Appendices B, C and D. Finally, Appendix E contains preliminary results of research into the behaviour of M-projections as the generating symmetry group becomes larger. The equation studied is the Boussinesq equation and the symmetry algebra which appears is isomorphic to $\mathfrak{sl}(3,\mathbb{R})$.

The achievements of the research embodied in this thesis can be grouped into three main categories.

Firstly, and most importantly, geometric structures based on symmetry groups have been established which will aid any future study into the role of transformations relating differential equations. Some indication of the potential of these structures is given in Section 3.4 (in constructing special solutions to differential equations), Sections 3.5 and 5.5 (in constructing auto-Bäcklund transformations for differential equations), Section 4.5 (in suggesting a possible Galois theory of Wahlquist-Estabrook prolongations of differential equations) and Chapter 6 (in identifying new algebraic structures associated with some integrable equations).

Secondly, a significant improvement of the construction of auto-Bäcklund transformations as invented by Wahlquist and Estabrook [92] and developed by Chen [10] has been uncovered. In common with those methods, a Wahlquist-Estabrook prolongation of the differential equation being studied must first be found, but the second step, identification of an auto-Bäcklund transformation, is considerably simplified. The complicated nonlinear conditions of Wahlquist and Estabrook and the rather unreliable search for discrete symmetries by Chen have been replaced by linear determining equations arising from the infinitesimal techniques involved.

Finally, another elegant algebraic structure has been added to the list of structures possessed by the KdV equation. This structure, the frequently occurring loop algebra over a finite-dimensional simple Lie algebra, is derived from the recursion operator of the KdV equation, which in turn arises from the bi-Hamiltonian formulation of that equation.

Some mention must be made of the important role of algebraic computing in the research presented here. Without the advent of symbolic manipulation packages, few, if any, of the results described here would have been discovered. The differential geometry package EXCALC [85] in REDUCE [39] was used to determine symmetry algebras and Wahlquist-Estabrook prolongations of differential equations, while general algebraic calculations were performed using MAPLE [9].

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Chapter 2

Preliminaries

This chapter describes most of the structures on which the constructions presented in subsequent chapters are based. The standard reference for the vast majority of this material is the book by Olver [72]. Wherever practicable, notation and terminology follow that standard. Those subjects not covered in [72], principally the prolongation method of Wahlquist and Estabrook, and those treated differently from [72] here, are covered in more detail than other topics in this chapter.

Some conventions regarding Lie groups and algebras, as well as local groups of transformations, are established in Section 2.1. The geometric structures which will be used are defined in Section 2.2 and are based on Olver's simplified version of jet bundles. The main point of departure from his approach is an increased reliance on contact forms [33]. In particular, the prolongation of group actions is performed by requiring invariance of the contact module [4] rather than Olver's use of representative functions. However, following Olver, attention is focused on "point" transformations — "contact" transformations are considered only briefly. Section 2.3 establishes definitions of differential equations and their symmetry groups. Differential equations and their solutions are interpreted in terms of submanifolds of jet spaces. The group reduction method for finding special solutions to differential equations is described in full, as it is used extensively later on. Olver's exposition of generalized symmetries (also known as Lie-Bäcklund transformations) is summarized in Section 2.4. Along with symmetry groups, the prolongation method invented by Wahlquist and Estabrook [23], [92] forms a key part of the foundations of this research. The prolongation method, which is presented in Section 2.5, is described in more detail than the other structures. It has been tailored to suit the applications the author has in mind as well as to make it compatible with the other structures used. Inevitably, some variations from existing versions of the prolongation method have resulted. However, it remains close in spirit to the "direct approach" favoured by Corones and Testa [15] and pursued by Nucci [68], [69]. Another suitable reference is the book by Pirani, Robinson and Shadwick [76].

2.1 Lie algebras

If G is a Lie group its Lie algebra will be denoted by the corresponding lower case German letter, $\mathfrak{g}$ in this case. Thus $\mathfrak{sl}(2,\mathbb{R})$ is the algebra of $SL(2,\mathbb{R})$, and so on. Lie algebras which arise frequently when studying many of the differential equations considered here are loop algebras. Let $\mathfrak{g}$ be a finite-dimensional Lie algebra with basis $\{X_a : a = 1,\ldots,r\}$ and Lie bracket

$$[X_a, X_b] = \sum_{c=1}^{r} C^c_{ab} X_c, \quad a, b = 1, \ldots, r,$$

with structure constants C^c_{ab}. The *loop algebra over* $\mathfrak{g}$, denoted $\mathfrak{g} \otimes \mathbb{R}[\lambda, \lambda^{-1}]$, is the Lie algebra with basis $\{X^m_a : a = 1,\ldots,r,\ m \in \mathbb{Z}\}$ and Lie bracket

$$[X^m_a, X^n_b] = \sum_{c=1}^{r} C^c_{ab} X^{m+n}_c, \quad a, b = 1, \ldots, r, \quad m, n \in \mathbb{Z}.$$

Loop algebras are usually only defined when $\mathfrak{g}$ is finite-dimensional and semi-simple, but the latter restriction is relaxed here. More information concerning loop algebras can be found in [14].

It will be assumed throughout that local groups of transformations G *act to the left* on M. That is, for all $g \in G$

$$g : x \mapsto g \cdot x$$

whenever $x \in M$ is in the domain of definition of g. If $g \cdot x$ is in the domain of definition of $h \in G$ and $(hg) \cdot x$ is defined, then

$$h \cdot (g \cdot x) = (hg) \cdot x.$$

It is further assumed that such group actions are *regular*, so that all G-orbits have the same dimension and for each point $x \in M$ there exist arbitrarily small neighbourhoods U of x with the property that each G-orbit intersects U in a pathwise connected set (Definition 1.26 of [72]). The action of G on M is *free* if the property $g \cdot x = x$ for all x in the domain of definition of $g \in G$ implies that $g = e$. Finally, if $\mathbf{v}$ is a vector field on M then $\exp(a\mathbf{v})$ will denote the one-parameter group of transformations generated by $\mathbf{v}$.

2.2 Geometric structures

An essential part of studying differential equations is being able to interpret them geometrically. Given a system of differential equations involving p independent and q dependent variables, the basic space on which all other structures are built is the Euclidean space $X \times U$, where $X = \mathbb{R}^p$ has coordinates $x = (x^1, \ldots, x^p)$, representing the independent variables, and $U = \mathbb{R}^q$ has coordinates $u = (u^1, \ldots, u^q)$, representing the dependent variables. For each $k = 1, 2, \ldots$ let

$$p_k = \binom{p+k-1}{k}$$

and introduce $U_k = \mathbb{R}^{qp_k}$ with coordinates u_J^α, where α ranges over $\{1, \ldots, q\}$ and $J = (j_1, \ldots, j_k)$ ranges over all unordered k-tuples of integers $j_l \in \{1, \ldots, p\}$. U_k represents the k-th order derivatives of functions $f : X \to U$. The Euclidean space $U^{(n)} = U \times U_1 \times \cdots \times U_n$ of dimension

$$q + qp_1 + \cdots + qp_n = q\binom{p+n}{n}$$

then represents all derivatives of order up to and including n. A typical point in $U^{(n)}$ will be denoted by $u^{(n)}$. It must be stressed that $u^{(n)}$ represents not just n-th order derivatives but also all derivatives up to order n. For instance, if X has coordinate x and U coordinate u then $u^{(3)}$ represents $(u, u_x, u_{xx}, u_{xxx})$. Returning to the general situation, $X \times U^{(n)}$ describes the space of independent and dependent variables together with k-th order derivatives for all $k \leq n$. When a differential equation is described on an open subset $M \subseteq X \times U$, the space $M^{(n)} = M \times U_1 \times \cdots \times U_n$ is called the *n-th order jet space*.

The principal advantage of the geometric structure above is that it allows all variables involved in a system of differential equations to be treated as coordinates in their own right. Thus, x^i, u^α and u_i^α are all treated on an equal footing. However, at some stage it will have to be recognized that u_i^α is not entirely independent of the other variables — it represents the derivative of u^α with respect to x^i. Such relationships are recorded using special differential one-forms on the jet space. For each $n = 1, 2, \ldots$ let $\Omega^{(n)}$ denote the module of forms on $M^{(n)}$ generated by the one-forms

$$\begin{aligned}
\theta^\alpha &= du^\alpha - \sum_{i=1}^{p} u_i^\alpha dx^i, \\
\theta_j^\alpha &= du_j^\alpha - \sum_{i=1}^{p} u_{ij}^\alpha dx^i, \\
&\vdots \\
\theta_{j_1 \cdots j_{n-1}}^\alpha &= du_{j_1 \cdots j_{n-1}}^\alpha - \sum_{i=1}^{p} u_{ij_1 \cdots j_{n-1}}^\alpha dx^i,
\end{aligned}$$

where α ranges over $\{1, \ldots, q\}$ and j_k over $\{1, \ldots, p\}$. Here it is important to remember that the coordinates on $U^{(k)}$ are labelled by unordered subscripts. That is, $u_{ij}^\alpha \equiv u_{ji}^\alpha$ and so on. These forms are called *contact forms* and $\Omega^{(n)}$ the *n-th order contact module.* Closely related to the contact module are *total derivatives.* If $f : M^{(n)} \to \mathbb{R}$ is a smooth function then the *total derivative of f with respect to x^i* is the smooth function $D_{x^i} f : M^{(n+1)} \to \mathbb{R}$ defined by

$$D_{x^i} f = \frac{\partial f}{\partial x^i} + \sum_{\alpha=1}^{q} \sum_{|J|=0}^{n} u_{iJ}^\alpha \frac{\partial f}{\partial u_J^\alpha}.$$

Here, if $J = (j_1, \ldots, j_k)$ then $|J|$ is defined to equal k.

Total derivatives and the contact module enable one to "prolong" many structures on M to analogous structures on any jet space $M^{(n)}$.

If a smooth mapping $f : Y \to U$, with Y an open subset of X, is described by $u^\alpha = f^\alpha(x)$, $\alpha = 1, \ldots, q$, then $\mathrm{pr}^{(n)} f : Y \to U^{(n)}$, the *n-th prolongation of f*, is the smooth function with components

$$u_{j_1 \cdots j_k}^\alpha = \frac{\partial^k f^\alpha}{\partial x^{j_1} \cdots \partial x^{j_k}}, \quad j_l = 1, \ldots, p, \quad \alpha = 1, \ldots, q, \quad k = 1, \ldots, n.$$

The action of a local group of transformations G on $M \subseteq X \times U$ can be prolonged to an action on any order jet space. For all nonnegative integers $k \leq n$ introduce the natural projections

$$\pi_k^n : M^{(n)} \to M^{(k)}, \quad (x, u^{(n)}) \mapsto (x, u^{(k)}).$$

The *n-th prolongation of* G, denoted $\mathrm{pr}^{(n)}G$, is the unique local group of transformations acting on $M^{(n)}$ such that

1. whenever $\mathrm{pr}^{(n)}g \cdot (x, u^{(n)})$ is defined,

$$\pi_k^n(\mathrm{pr}^{(n)}g \cdot (x, u^{(n)})) = \mathrm{pr}^{(k)}g \cdot (x, u^{(k)}),$$

   for all nonnegative integers $k \leq n$, and

2. $(\mathrm{pr}^{(n)}g)^*\Omega^{(n)} \subseteq \Omega^{(n)}$ for all $g \in G$.

Property 1 amounts to the following diagram commuting:

$$\begin{array}{ccc} M^{(n)} & \xrightarrow{\mathrm{pr}^{(n)}g} & M^{(n)} \\ \pi_k^n \big\downarrow & & \big\downarrow \pi_k^n \\ M^{(k)} & \xrightarrow{\mathrm{pr}^{(k)}g} & M^{(k)} \end{array}$$

The prolongation of a group action is constructed recursively. If

$$\mathrm{pr}^{(n)}g \cdot (x, u^{(n)}) = (y(x, u^{(n)}), v^{(n)}(x, u^{(n)})) \tag{2.1}$$

then property 1 indicates that all components of $\mathrm{pr}^{(n+1)}g \cdot (x, u^{(n+1)})$ agree with those of $\mathrm{pr}^{(n)}g \cdot (x, u^{(n)})$ in equation (2.1), with the exception of v_{iJ}^α for $|J| = n$. These remaining components are completely determined by property 2. Invariance of the contact module under the action of $\mathrm{pr}^{(n)}g$ requires that v_{iJ}^α satisfy the system of linear algebraic equations

$$\sum_{i=1}^{p} \left(D_{x^j} y^i\right) v_{iJ}^\alpha = D_{x^j} v_J^\alpha, \quad j = 1, \ldots, p, \quad \alpha = 1, \ldots, q, \quad |J| = n. \tag{2.2}$$

When g is sufficiently close to the identity this system has a unique solution, yielding $\mathrm{pr}^{(n+1)}g \cdot (x, u^{(n+1)})$. The local group of transformations $\mathrm{pr}^{(n)}G$ is sometimes called the prolongation of the local group of *point* transformations G.

Construction of the prolonged group action, as described by equations (2.2), can become horribly complicated. The infinitesimal generators of the G- and $\mathrm{pr}^{(n)}G$-actions, on M and $M^{(n)}$ respectively, are much simpler to work with. If

$$\mathbf{v} = \sum_{i=1}^{p} \xi^i(x, u)\partial_{x^i} + \sum_{\alpha=1}^{q} \phi^\alpha(x, u)\partial_{u^\alpha}$$

is an infinitesimal generator of the local group of transformations G acting on M then

$$\mathrm{pr}^{(n)}\mathbf{v} = \sum_{i=1}^{p} \xi^i(x,u)\partial_{x^i} + \sum_{\alpha=1}^{q} \sum_{|J|=0}^{n} \phi_J^\alpha(x,u^{(n)})\partial_{u_J^\alpha}$$

is an infinitesimal generator of the local group of transformations $\mathrm{pr}^{(n)}G$ acting on $M^{(n)}$, where the functions ϕ_J^α are given recursively by

$$\phi_{Jk}^\alpha = D_{x^k}\phi_J^\alpha - \sum_{i=1}^{p} u_{iJ}^\alpha D_{x^k}\xi^i. \tag{2.3}$$

For completeness, a generalization of point transformations is included. These transformations will not be used in the body of this work, although they make a brief appearance in Appendix A in connection with the research of other authors. A local group of transformations G acting on $M^{(1)}$ determines a local group of *contact transformations* if and only if $g^*\Omega^{(1)} \subseteq \Omega^{(1)}$ for all $g \in G$. If

$$g \cdot (x,u^{(1)}) = (y(x,u^{(1)}), v^{(1)}(x,u^{(1)}))$$

then the requirement that

$$g^*\left(dv^\alpha - \sum_{i=1}^{p} v_i^\alpha dy^i\right) \in \Omega^{(1)}, \quad \alpha = 1,\ldots,q,$$

is equivalent to the system of differential equations

$$\frac{\partial v^\alpha}{\partial u_j^\beta} = \sum_{i=1}^{p} v_i^\alpha \frac{\partial y^i}{\partial u_j^\beta}, \quad j=1,\ldots,p, \quad \alpha,\beta = 1,\ldots,q,$$

$$\frac{\partial v^\alpha}{\partial x^j} + \sum_{\beta=1}^{q} u_j^\beta \frac{\partial v^\alpha}{\partial u^\beta} = \sum_{i=1}^{p} v_i^\alpha \left(\frac{\partial y^i}{\partial x^j} + \sum_{\beta=1}^{q} u_j^\beta \frac{\partial y^i}{\partial u^\beta}\right), \quad j=1,\ldots,p, \quad \alpha=1,\ldots,q,$$

being satisfied [4]. As with point transformations, contact transformations can be prolonged to local groups of transformations acting on higher order jet spaces. The *n-th prolongation of* G, denoted $\mathrm{pr}^{(n)}G$, is the unique local group of transformations acting on $M^{(n+1)}$ such that

1. whenever $\mathrm{pr}^{(n)}g \cdot (x,u^{(n+1)})$ is defined,

$$\pi_{k+1}^{n+1}(\mathrm{pr}^{(n)}g \cdot (x,u^{(n+1)})) = \mathrm{pr}^{(k)}g \cdot (x,u^{(k+1)}),$$

for all nonnegative integers $k \leq n$, and

2. $(\mathrm{pr}^{(n)}g)^*\Omega^{(n+1)} \subseteq \Omega^{(n+1)}$ for all $g \in G$.

The second requirement can be expressed in terms of a system of equations, analogous to equations (2.2), which is not presented here. Details, and properties of the infinitesimal generators of local groups of contact transformations can be found in Section 5.2.4 of [4].

2.3 Differential equations and symmetries

The geometric structures introduced above can be utilized in the study of differential equations and their symmetry properties. A *system of n-th order differential equations* Δ involving p independent and q dependent variables is a system of equations

$$\Delta^l(x, u^{(n)}) = 0, \quad l = 1, \dots, m,$$

where $x = (x^1, \dots, x^p)$ and $u = (u^1, \dots, u^q)$ are coordinates on $X = \mathbb{R}^p$ and $U = \mathbb{R}^q$ respectively. This notation will be used throughout the current section. Assuming that each Δ^l is smooth in its arguments, this system of equations leads to a smooth mapping $\Delta : M^{(n)} \to \mathbb{R}^m$ and determines a subvariety $\mathcal{S}_\Delta = \ker \Delta$ of the n-th order jet space. Δ is said to have *maximal rank* if the Jacobian matrix

$$\mathrm{J}_\Delta(x, u^{(n)}) = \left(\frac{\partial \Delta^l}{\partial x^i}, \frac{\partial \Delta^l}{\partial u^\alpha_J} \right)$$

has rank m for all $(x, u^{(n)}) \in \mathcal{S}_\Delta$. It will be assumed throughout that the maximal rank condition is satisfied by the differential equations considered.

A *solution* to Δ is any p-dimensional submanifold $\Phi : N \to M^{(n)}$ which satisfies $\Phi(N) \subseteq \mathcal{S}_\Delta$ and $\Phi^*\Omega^{(n)} = 0$. The relationship between this notion of solution and the more common one is given in the following lemma.

Lemma 2.1 *Let Δ be a system of n-th order differential equations on $M \subseteq X \times U$ and suppose that $\Phi : N \to M^{(n)}$ is a solution to Δ. If $f : Y \to U$ is a smooth mapping, with Y an open subset of X, and*

$$\{(x, f(x)) : x \in Y\} \subseteq \pi^n_0(\Phi(N))$$

then

$$\Delta^l(x, \mathrm{pr}^{(n)} f(x)) = 0, \quad l = 1, \dots, m, \tag{2.4}$$

for all $x \in Y$.

PROOF: Let N have coordinates $y = (y^1, \ldots, y^p)$ and suppose that Φ is described by

$$\Phi : y \mapsto (x(y), u^{(n)}(y)).$$

Introduce the open subset $\tilde{N} \subseteq N$ containing those points y such that $x(y) \in Y$. Then for all $y \in \tilde{N}$

$$u^\alpha(y) = f^\alpha(x) = f^\alpha \circ x(y), \quad \alpha = 1, \ldots, q,$$

and

$$\frac{\partial u^\alpha}{\partial y^i} = \sum_{j=1}^{p} \frac{\partial f^\alpha}{\partial x^j} \frac{\partial x^j}{\partial y^i}, \quad i = 1, \ldots, p, \quad \alpha = 1, \ldots, q,$$

so that

$$\begin{aligned} 0 = \Phi^* \theta^\alpha &= \sum_{i=1}^{p} \left(\frac{\partial u^\alpha}{\partial y^i} - \sum_{j=1}^{p} u^\alpha_j \frac{\partial x^j}{\partial y^i} \right) dy^i, \\ &= \sum_{i,j=1}^{p} \left(\frac{\partial f^\alpha}{\partial x^j} - u^\alpha_j \right) \frac{\partial x^j}{\partial y^i} dy^i, \quad \alpha = 1, \ldots, q. \end{aligned}$$

On $\tilde{N}$ the matrix $(\frac{\partial x^j}{\partial y^i})$ has rank p, forcing

$$u^\alpha_i(y) = \frac{\partial f^\alpha}{\partial x^i}(x), \quad i = 1, \ldots, p, \quad \alpha = 1, \ldots, q,$$

for all $y \in \tilde{N}$ with $x = x(y) \in Y$. Therefore

$$\{(x, \mathrm{pr}^{(1)} f(x)) : x \in Y\} \subseteq \pi^n_1 \circ \Phi(N)$$

and continuing in this manner will show that

$$\{(x, \mathrm{pr}^{(n)} f(x)) : x \in Y\} \subseteq \Phi(N) \subseteq \mathcal{S}_\Delta,$$

which completes the proof. □

The classical definition of a solution to Δ is any smooth mapping $f : Y \to U$ satisfying equation (2.4). Given a solution $\Phi : N \to M^{(n)}$ satisfying the definition used here, Lemma 2.1 shows that if the corresponding submanifold of M contains the "graph" of a function $f : Y \to U$ then f must be a solution to Δ in the classical sense. Olver used a similar notion of solution to that adopted here in his presentation of group-invariant solution methods using "extended jet spaces" (see Section 3.5 of [72]).

Suppose that $N \subseteq \mathbb{R}^p$ has coordinates $y = (y^1, \ldots, y^p)$ and that

$$\Phi : N \to M^{(n)}, \quad y \mapsto (x(y), u^{(n)}(y)),$$

is a solution to Δ such that the matrix $(\dfrac{\partial x^j}{\partial y^i})$ has rank p on N. Then for each nonnegative integer k there exists a unique submanifold $\mathrm{pr}^{(k)}\Phi : N \to M^{(n+k)}$, called the *$k$-th prolongation of* Φ, such that

1. $\pi_n^{n+k} \circ \mathrm{pr}^{(k)}\Phi = \Phi$ on N and
2. $(\mathrm{pr}^{(k)}\Phi)^*\theta_J^\alpha = 0$ for all $\alpha = 1, \ldots, q$ and all multi-indices J with $|J| = n, \ldots, n+k-1$.

This submanifold is found by recursively solving the systems of equations

$$\sum_{i=1}^{p} \frac{\partial x^i}{\partial y^j} u_{Ji}^\alpha = \frac{\partial u_J^\alpha}{\partial y^j}, \quad j = 1, \ldots, p, \quad \alpha = 1, \ldots, q, \quad |J| = n, \ldots, n+k-1,$$

which arise from the requirement that $(\mathrm{pr}^{(k)}\Phi)^*\theta_J^\alpha = 0$. Each $\mathrm{pr}^{(k)}\Phi : N \to M^{(n+k)}$ is a solution to the *k-th prolongation of* Δ, which is defined to be the system of $(n+k)$-th order differential equations

$$D_{x^{i_1}} \cdots D_{x^{i_j}} \Delta^l(x, u^{(n+k)}) = 0, \quad j = 0, \ldots, k, \quad 1 \le i_1 \le \cdots \le i_j \le p, \quad l = 1, \ldots, m,$$

comprising Δ and the *differential consequences* obtained by evaluating all j-th order total derivatives of Δ for $j = 0, \ldots, k$. For instance, $(\mathrm{pr}^{(1)}\Phi)^*\Omega^{(n+1)} = 0$ by the definition of $\mathrm{pr}^{(1)}\Phi$ and, since $(\dfrac{\partial x^j}{\partial y^i})$ is invertible, the equations

$$\begin{aligned}
\sum_{i=1}^{p} \frac{\partial x^i}{\partial y^j}(D_{x^i}\Delta^l) &= \sum_{i=1}^{p} \frac{\partial x^i}{\partial y^j}\left(\frac{\partial \Delta^l}{\partial x^i} + \sum_{\alpha=1}^{q}\sum_{|J|=0}^{n} u_{Ji}^\alpha \frac{\partial \Delta^l}{\partial u_J^\alpha}\right) \\
&= \sum_{i=1}^{p} \frac{\partial x^i}{\partial y^j}\frac{\partial \Delta^l}{\partial x^i} + \sum_{\alpha=1}^{q}\sum_{|J|=0}^{n} \frac{\partial u_J^\alpha}{\partial y^j}\frac{\partial \Delta^l}{\partial u_J^\alpha} \\
&= \frac{\partial \Delta^l}{\partial y^j}, \quad l = 1, \ldots, m, \quad j = 1, \ldots, p,
\end{aligned}$$

imply that $D_{x^i}\Delta^l = 0$ on $\mathrm{pr}^{(1)}\Phi(N)$. Since Δ^l certainly vanishes on $\mathrm{pr}^{(1)}\Phi(N)$, it follows that $\mathrm{pr}^{(1)}\Phi$ is a solution to the first prolongation of Δ.

A (classical) *symmetry group* G of the differential equation Δ is a local group of transformations acting on M such that if $\Phi : N \to M^{(n)}$ is a solution to Δ and $\Phi(N)$ is in the domain of definition of $\mathrm{pr}^{(n)}g$ for some $g \in G$ then the submanifold

$$\mathrm{pr}^{(n)}g \cdot \Phi : N \to M^{(n)}, \quad y \mapsto \mathrm{pr}^{(n)}g \cdot \Phi(y),$$

is also a solution to Δ. The differential equation Δ is said to be *locally solvable* if given any point $(x_0, u_0^{(n)}) \in \mathcal{S}_\Delta$ there is a solution to Δ passing through this point (see Section 2.6 of [72]). It will be assumed throughout this work that all differential equations are locally solvable. Thus, a local group of transformations G acting on M is a symmetry group of Δ if and only if whenever $(x, u^{(n)}) \in \mathcal{S}_\Delta$ it follows that $\mathrm{pr}^{(n)}g \cdot (x, u^{(n)}) \in \mathcal{S}_\Delta$ for all $g \in G$ such that this is defined (Theorem 2.71 of [72]).

The maximal rank condition defined above allows one to determine all connected symmetry groups of a system of differential equations using infinitesimal techniques. Suppose that Δ has maximal rank and that G is a local group of transformations acting on M. Using Theorem 2.8 of [72], G is a symmetry group of Δ if and only if the functions

$$\mathrm{pr}^{(n)}\mathbf{v}(\Delta^l) : M^{(n)} \to \mathbb{R}, \quad l = 1, \ldots, m,$$

vanish identically on $\mathcal{S}_\Delta$ for every infinitesimal generator $\mathbf{v}$ of G. The prolongation formula, equation (2.3), together with the requirement $\mathrm{pr}^{(n)}\mathbf{v}(\Delta^l)(\mathcal{S}_\Delta) = 0$ combine to give a system of linear partial differential equations for the functions $\{\xi^i, \phi^\alpha : i = 1, \ldots, p, \ \ \alpha = 1, \ldots, q\}$ appearing in

$$\mathbf{v} = \sum_{i=1}^{p} \xi^i(x,u)\partial_{x^i} + \sum_{\alpha=1}^{q} \phi^\alpha(x,u)\partial_{u^\alpha}.$$

These equations can often be solved systematically, yielding the most general connected symmetry group of Δ satisfying the definition used here. Examples 2.41 to 2.45 of [72] demonstrate the construction of symmetry groups for some well known differential equations.

Let G be a local group of transformations acting on M generated by $\{\mathbf{v}_a : a = 1, \ldots, r\}$, where

$$\mathbf{v}_a = \sum_{i=1}^{p} \xi_a^i(x,u)\partial_{x^i} + \sum_{\alpha=1}^{q} \phi_a^\alpha(x,u)\partial_{u^\alpha}, \quad a = 1, \ldots, r.$$

If the subvariety of $M^{(n)}$ determined by the equations $\{\Delta^l = 0 : l = 1, \ldots, m\}$,

$$\phi_a^\alpha - \sum_{i=1}^{p} u_i^\alpha \xi_a^i = 0, \quad \alpha = 1, \ldots, q, \quad a = 1, \ldots, r, \tag{2.5}$$

and all k-th order total derivatives of equations (2.5) with $k \le n-1$ is invariant under the action of $\mathrm{pr}^{(n)}G$, then G is called a *conditional symmetry group* of Δ [63]. The equations determining the infinitesimal generators of conditional symmetry groups are generally nonlinear, so that conditional symmetry groups are more difficult to construct systematically than (classical) symmetry groups. Notice that all (classical) symmetry groups are certainly conditional symmetry groups.

Let G be any local group of transformations acting on M. A submanifold $\Phi : N \to M^{(n)}$ is called a *G-invariant solution of* Δ if it is a solution of Δ and is *locally* $\mathrm{pr}^{(n)}G$*-invariant* as a submanifold of $M^{(n)}$. That is, for each $y \in N$ there exists a neighbourhood of the identity in G such that $\mathrm{pr}^{(n)}g \cdot (\Phi(y)) \in \Phi(N)$ for all g in that neighbourhood. G-invariant solutions of Δ are constructed as follows:

1. Assuming that G acts freely and regularly on M, it has orbits of dimension $s = \dim G$. This assumption allows the construction of $p + q - s$ functionally independent invariants of the G-action on M. Let these invariants be

$$\begin{aligned} y^i &= \eta^i(x, u), \quad i = 1, \ldots, p - s, \\ v^\alpha &= \zeta^\alpha(x, u), \quad \alpha = 1, \ldots, q, \end{aligned} \tag{2.6}$$

where the grouping into y and v invariants is entirely arbitrary.

2. Provided that the matrix

$$\begin{pmatrix} \dfrac{\partial \eta^i}{\partial u^\beta} \\ \dfrac{\partial \zeta^\alpha}{\partial u^\beta} \end{pmatrix}$$

has rank q everywhere, equations (2.6) can be solved for $p-s$ of the x variables, denoted by $\tilde{x}$, and all of the u-variables in terms of y, v and the s remaining x-variables, denoted by $\hat{x}$.

3. Using the chain rule, x-derivatives of u can be written as functions of $\hat{x}$, y, v and y-derivatives of v.

4. After substituting in these expressions, the differential equation Δ can be simplified to an equation involving $v(y)$ and $\hat{x}$. When G is a conditional symmetry group of Δ the $\hat{x}$-dependence can be eliminated, leaving a differential equation for $v(y)$. This equation, called the *G-reduction of* Δ, involves just $p-s$ independent variables, $y=(y^1,\ldots,y^{p-s})$.

In this process, known as the *group reduction method*, y and v are called *principal variables* and $\hat{x}$ are known as *parametric variables.* When G is a conditional symmetry group, this reduction method corresponds to the "nonclassical method" of Bluman and Cole [3] for finding special solutions to differential equations. The group reduction method, restricted to the case where G is a (classical) symmetry group, is presented rigorously in Section 3.5 of [72] and extended in [74]. Examples 3.3 to 3.5 of [72] demonstrate this technique by constructing group-invariant solutions of some well known differential equations.

2.4 Generalized symmetries

Let $M\subseteq X\times U$ be an open subset with $X=\mathbb{R}^p$ and $U=\mathbb{R}^q$ having coordinates $x=(x^1,\ldots,x^p)$ and $u=(u^1,\ldots,u^q)$ respectively. The algebra of smooth real-valued functions of $(x,u^{(n)})$ for arbitrary finite n is denoted by $\mathfrak{A}$. Elements of $\mathfrak{A}$ are called *differential functions* and denoted by $P[u]$. $\mathfrak{A}^m$ denotes the vector space of m-tuples $P[u]=(P^1[u],\ldots,P^m[u])^T$ of elements in $\mathfrak{A}$. Of special interest are certain linear operators on $\mathfrak{A}$. For each $i=1,\ldots,p$, the *total derivative operator with respect to* x^i is the linear operator

$$D_{x^i}:\mathfrak{A}\to\mathfrak{A},\quad P[u]\mapsto\frac{\partial P}{\partial x^i}+\sum_{\alpha=1}^{q}\sum_{|J|=0}^{\infty}u^\alpha_{iJ}\frac{\partial P}{\partial u^\alpha_J}.$$

Of course, each $P[u]\in\mathfrak{A}$ can be identified with a smooth function $P:M^{(n)}\to\mathbb{R}$ for some finite n. The total derivative of this function with respect to x^i, $D_{x^i}P:M^{(n+1)}\to\mathbb{R}$, which was defined earlier can be identified with $D_{x^i}(P[u])\in\mathfrak{A}$, so that total derivative operators and total derivatives of smooth real-valued functions on some jet space can be used interchangeably. Generally, total derivative operators are preferred when the order of the jet space involved is unknown or unimportant.

The formal expression

$$\mathbf{v} = \sum_{i=1}^{p} \xi^i[u]\partial_{x^i} + \sum_{\alpha=1}^{q} \phi^\alpha[u]\partial_{u^\alpha} \tag{2.7}$$

is called a *generalized vector field* on M whenever $\xi^i[u]$ and $\phi^\alpha[u]$ are differential functions. Generalized vector fields prolong in a manner almost identical to that of the (geometrical) vector fields described in Section 2.2. For each nonnegative integer n, the *n-th prolongation of* $\mathbf{v}$ is the generalized vector field

$$\mathrm{pr}^{(n)}\mathbf{v} = \sum_{i=1}^{p} \xi^i[u]\partial_{x^i} + \sum_{\alpha=1}^{q}\sum_{|J|=0}^{n} \phi^\alpha_J[u]\partial_{u^\alpha_J},$$

where the differential functions $\phi^\alpha_J[u] \in \mathfrak{A}$ are defined recursively by

$$\phi^\alpha_{Jk}[u] = D_{x^k}\left(\phi^\alpha_J[u]\right) - \sum_{i=1}^{p} u^\alpha_{iJ} D_{x^k}(\xi^i[u]).$$

Generalized vector fields of the form

$$\mathbf{v}_Q = \sum_{\alpha=1}^{q} Q^\alpha[u]\partial_{u^\alpha} \tag{2.8}$$

are called *evolutionary vector fields* and are said to have *characteristic*

$$Q[u] = (Q^1[u], \ldots, Q^q[u])^T \in \mathfrak{A}^q.$$

Evolutionary vector fields are especially easy to prolong since

$$\mathrm{pr}^{(n)}\mathbf{v}_Q = \sum_{\alpha=1}^{q}\sum_{|J|=0}^{n} D_J\left(Q^\alpha[u]\right)\partial_{u^\alpha_J}$$

for each nonnegative integer n. Here, and throughout, $D_J \equiv D_{x^{j_1}} \cdots D_{x^{j_r}}$ whenever $J = (j_1, \ldots, j_r)$. Any generalized vector field has a unique *evolutionary representative.* The evolutionary representative of $\mathbf{v}$ in equation (2.7) is $\mathbf{v}_Q$ in equation (2.8) with

$$Q^\alpha[u] = \phi^\alpha[u] - \sum_{i=1}^{p} u^\alpha_i \xi^i[u], \quad \alpha = 1, \ldots, q.$$

The *characteristic* of a generalized vector field is the characteristic of its evolutionary representative.

A *generalized symmetry generator* of a system Δ of n-th order differential equations on M is any generalized vector field $\mathbf{v}$ on M such that

$$\mathrm{pr}^{(n)}\mathbf{v}(\Delta^l) = \sum_{k=1}^{m} \mathcal{D}^l_k(\Delta^k), \quad l = 1, \ldots, m,$$

for some differential operators $\mathcal{D}_k^l = \sum_{|J|=0}^{\infty} P_k^{lJ}[u] D_J$ on $\mathfrak{A}$. Straightforward calculations prove that $\mathbf{v}$ is a generalized symmetry of Δ if and only if its evolutionary representative $\mathbf{v}_Q$ is a generalized symmetry generator (Proposition 5.5 of [72]). This result, together with the ease with which evolutionary vector fields can be prolonged explains the importance of those vector fields.

Evolutionary vector fields also yield an attractive characterization of generalized symmetries. The *Fréchet derivative* of $P[u] \in \mathfrak{A}^m$ is the $m \times q$ matrix D_P with entries

$$(\mathsf{D}_P)^l_\alpha = \sum_{|J|=0}^{\infty} \frac{\partial P^l}{\partial u^\alpha_J} D_J, \quad \alpha = 1, \ldots, q, \;\; l = 1, \ldots, m.$$

If $P[u]$ depends only on $(x, u^{(n)})$ then it can be shown that

$$\sum_{\alpha=1}^{q} (\mathsf{D}_P)^l_\alpha (Q^\alpha) = \mathrm{pr}^{(n)} \mathbf{v}_Q(P^l), \quad l = 1, \ldots, m,$$

for all $Q \in \mathfrak{A}^q$ (Proposition 5.29 of [72]). Therefore, an evolutionary vector field $\mathbf{v}_Q$ is a generalized symmetry of Δ if and only if

$$\sum_{\alpha=1}^{q} (\mathsf{D}_\Delta)^l_\alpha (Q^\alpha) = \mathrm{pr}^{(n)} \mathbf{v}_Q(\Delta^l) = \sum_{k=1}^{m} \mathcal{D}^l_k(\Delta^k), \quad l = 1, \ldots, m,$$

for some differential operators $\mathcal{D}_k^l = \sum_{|J|=0}^{\infty} P_k^{lJ}[u] D_J$ on $\mathfrak{A}$.

2.5 Wahlquist-Estabrook prolongations

Let Δ denote a system of n-th order differential equations on $M \subseteq X \times U$ where $X = \mathbb{R}^p$ and $U = \mathbb{R}^q$ have coordinates $x = (x^1, \ldots, x^p)$ and $u = (u^1, \ldots, u^q)$ respectively. Introduce the space $Y = \mathbb{R}^r$ with r arbitrary and $(y^1, \ldots, y^r)$ as coordinates. If the smooth functions $\{F^a_i(x, u^{(n-1)}, y) : i = 1, \ldots, p, \;\; a = 1, \ldots, r\}$ are such that

$$\left(D_{x^j} F^a_i - D_{x^i} F^a_j - \sum_{b=1}^{r} \left(F^b_i \frac{\partial F^a_j}{\partial y^b} - F^b_j \frac{\partial F^a_i}{\partial y^b} \right) \right) (x, u^{(n)}, y) = 0 \qquad (2.9)$$

for all $i, j = 1, \ldots, p$ and $a = 1, \ldots, r$ whenever $(x, u^{(n)}) \in \mathcal{S}_\Delta$, then the equations

$$0 = \Xi^a_i = y^a_i - F^a_i(x, u^{(n-1)}, y), \quad i = 1, \ldots, p, \;\; a = 1, \ldots, r, \qquad (2.10)$$

are said to define a *Wahlquist-Estabrook prolongation* (Δ, Ξ) *of* Δ to $M \times Y$ [92]. The new dependent variables $y^1, \ldots, y^r$ are called *pseudopotentials*, Y is known as the *pseudopotential space* and equations (2.10) are termed *prolongation equations*. For the expression $D_{x^j}F_i^a$ to make sense, F_i^a is treated as a function mapping $M^{(n-1)}$ to $\mathbb{R}$ with $y^1, \dots, y^r$ appearing as parameters only. The Wahlquist-Estabrook prolongation process converts a system of differential equations Δ into a larger system (Δ, Ξ) involving more equations and more dependent variables. Equations (2.9) are important as they ensure that the integrability conditions among the additional equations do not impose restrictions on the original equations. That is, equations such as

$$\frac{\partial^2 y^a}{\partial x^i \partial x^j} = \frac{\partial^2 y^a}{\partial x^j \partial x^i}, \quad i, j = 1, \dots, p, \quad a = 1, \dots, r,$$

contribute no extra equations to the system. Consequently, if a solution to Δ is given in the form $u = f(x)$ then the differential equations

$$\frac{\partial y^a}{\partial x^i} = F_i^a(x, \mathrm{pr}^{(n-1)} f(x), y), \quad i = 1, \dots, p, \quad a = 1, \dots, r,$$

for $y(x)$ are *completely integrable*. This observation has a geometric interpretation in terms of the concept of solutions used here. If $\Phi : N \to M^{(n)}$ is a solution of Δ then the submanifold

$$\tilde{\Phi} : N \times Y \to M^{(n)} \times Y, \quad (z, y) \mapsto (\Phi(z), y),$$

admits a p-dimensional foliation into solutions of (Δ, Ξ). The foliation is determined by the ideal of differential forms on $\tilde{\Phi}(N \times Y) = \Phi(N) \times Y$ generated by

$$\omega^a = dy^a - \sum_{i=1}^{p} F_i^a(x, u^{(n-1)}, y) dx^i, \quad a = 1, \dots, r,$$

which is closed under exterior differentiation due to equations (2.9). The one-forms ω^a correspond to the *prolongation forms* introduced by Wahlquist and Estabrook. A useful introduction to foliations is contained in [59].

Due to the nonlinear nature of equations (2.9), the construction of Wahlquist-Estabrook prolongations of a given differential equation can become quite complicated. The next example considers this problem when the differential equation involved is Burgers' equation.

Example 2.2 Burgers' equation $u_t = u_{xx} + 2uu_x$ is described on $M = X \times U$ where $X = \mathbb{R}^2$ and $U = \mathbb{R}^1$ have coordinates (x, t) and u respectively. The equations determining Wahlquist-Estabrook prolongations of Burgers' equation are thus

$$0 = D_t F^a - D_x G^a - \sum_{b=1}^{r} \left(F^b \frac{\partial G^a}{\partial y^b} - G^b \frac{\partial F^a}{\partial y^b} \right), \quad a = 1, \ldots, r, \tag{2.11}$$

whenever $u_t = u_{xx} + 2uu_x$. The corresponding prolongation equations will be

$$y^a_x = F^a(x, t, u, u_x, u_t, y), \quad y^a_t = G^a(x, t, u, u_x, u_t, y), \quad a = 1, \ldots, r,$$

where the precise value of r is still to be determined. By replacing every appearance of u_{xx} in equations (2.11) with $u_t - 2uu_x$ one obtains the system of equations

$$\begin{aligned} 0 &= F^a_t + u_t F^a_u - G^a_x - u_x G^a_u + (2uu_x - u_t) G^a_{u_x} \\ &\quad + u_{xt}(F^a_{u_x} - G^a_{u_t}) + u_{tt} F^a_{u_t} - \sum_{b=1}^{r} \left(F^b \frac{\partial G^a}{\partial y^b} - G^b \frac{\partial F^a}{\partial y^b} \right) \end{aligned} \tag{2.12}$$

for $\{F^a, G^a : a = 1, \ldots, r\}$. Derivatives are described using subscript notation. From the coefficient of u_{tt}, $F^a = F^a(x, t, u, u_x, y)$ and then the coefficient of u_{xt} implies that $G^a = u_t F^a_{u_x} + H^a(x, t, u, u_x, y)$, completely determining the u_t-dependence of $\{F^a, G^a\}$. Therefore coefficients of various powers of u_t in equations (2.12) can be equated to zero. For instance, from the coefficient of u_t^2,

$$F^a = I^a(x, t, u, y) + u_x J^a(x, t, u, y), \quad a = 1, \ldots, r,$$

and from that of u_t,

$$H^a = K^a(x, t, u, y) + u_x L^a(x, t, u, y), \quad a = 1, \ldots, r.$$

At this stage, the u_x-dependence of $\{F^a, G^a\}$ is specified and equations (2.12), when rewritten as polynomials in u_x and u_t, yield a system of equations determining the remaining functions $\{I^a, J^a, K^a, L^a\}$. Introducing the notation

$$[M, N]^a = \sum_{b=1}^{r} \left(M^b \frac{\partial N^a}{\partial y^b} - N^b \frac{\partial M^a}{\partial y^b} \right), \quad a = 1, \ldots, r,$$

this system can be written as

$$\begin{aligned} 0 &= I^a_u - J^a_x - L^a - [I, J]^a, \\ 0 &= L^a_u + [J, L]^a, \\ 0 &= 2uL^a + J^a_t - K^a_u - L^a_x - [I, L]^a - [J, K]^a, \\ 0 &= I^a_t - K^a_x - [I, K]^a. \end{aligned} \tag{2.13}$$

Every solution $\{I^a, J^a, K^a, L^a : a = 1, \ldots, r\}$ to this system yields a Wahlquist-Estabrook prolongation of Burgers' equation described by

$$y_x^a = I^a + u_x J^a, \quad y_t^a = u_t J^a + K^a + u_x L^a.$$

Traditionally, equations (2.13) are solved after making some simplifying assumptions. Suppose, for instance, that $J^a = 0$ and the functions $\{I^a, K^a, L^a : a = 1, \ldots, r\}$ are independent of x and t. One immediately finds that $L^a = X_1^a(y)$, for some functions X_1^a of the pseudopotentials only, and then that $I^a = uX_1^a(y) + X_2^a(y)$, for some functions X_2^a. The third equation of equations (2.13) then forces

$$K^a = u^2 X_1^a(y) + uX_3^a(y) + X_4^a(y)$$

with X_3^a and X_4^a still to be determined. Substituting this expression into equations (2.13) and treating the result as an identity in u yields the system of equations

$$\begin{aligned} 0 &= [X_1, X_2]^a - X_3^a, \\ 0 &= [X_1, X_3]^a - X_3^a, \\ 0 &= [X_1, X_4]^a + [X_2, X_3]^a, \\ 0 &= [X_2, X_4]^a. \end{aligned} \tag{2.14}$$

The equations

$$\begin{aligned} y_x^a &= uX_1^a(y) + X_2^a(y), \\ y_t^a &= (u_x + u^2)X_1^a(y) + uX_3^a(y) + X_4^a(y), \end{aligned} \tag{2.15}$$

describe a Wahlquist-Estabrook prolongation of Burgers' equation whenever the functions $\{X_1^a, X_2^a, X_3^a, X_4^a : a = 1, \ldots, r\}$ satisfy equations (2.14). This system has been obtained and studied by Dodd and Fordy [18] and Finley and McIver [24]. □

Systems such as equations (2.15) are sufficiently important to merit special attention. Suppose that one substitutes the *Ansatz*

$$F_i^a = \sum_{\mu=1}^{s} \sigma_i^\mu(x, u^{(n-1)}) X_\mu^a(y), \quad i = 1, \ldots, p, \ \ a = 1, \ldots, r, \tag{2.16}$$

with $\{\sigma_i^\mu : i = 1, \ldots, p, \ \mu = 1, \ldots, s\}$ fixed smooth functions on $M^{(n-1)}$, into the equations defining a Wahlquist-Estabrook prolongation, equations (2.9). Restricting the resulting system to $\mathcal{S}_\Delta \times Y$ and treating the result as an identity will lead to a system of differential equations for the unknown functions $\{X^a_\mu : a = 1, \ldots, r,\ \mu = 1, \ldots, s\}$. Each equation in this system will have the form

$$0 = \sum_{\mu,\nu=1}^{s} A^{\mu\nu}[X_\mu, X_\nu]^a + \sum_{\mu=1}^{s} B^\mu X^a_\mu, \quad a = 1, \ldots, r, \tag{2.17}$$

for suitable *constants* $A^{\mu\nu}$, B^μ. The *Wahlquist-Estabrook prolongation algebra* $\mathcal{L}$ associated with the *Ansatz* in equations (2.16) is defined to be the free Lie algebra generated by elements $\{X_\mu : \mu = 1, \ldots, s\}$ modulo the commutator relationships

$$0 = \sum_{\mu,\nu=1}^{s} A^{\mu\nu}[X_\mu, X_\nu] + \sum_{\mu=1}^{s} B^\mu X_\mu$$

corresponding to each set of differential equations given by equations (2.17) [86]. Of course, if one is to obtain a nontrivial prolongation algebra then the *Ansatz* of equations (2.16) must be chosen carefully. The process demonstrated in Example 2.2, where one first looks for a general prolongation before imposing certain constraints to obtain expressions of the form given in equations (2.15), is typical. It is important to appreciate that the prolongation algebra depends on the form of equations (2.16) and may change with different *Ansätze*. One feature of the prolongation algebra is that it converts the problem of solving the differential equations (2.17) into one of finding representations of a Lie algebra. In particular, the functions $\{X^a_\mu : a = 1, \ldots, r,\ \ \mu = 1, \ldots, s\}$ satisfy equations (2.17) if and only if the mapping

$$\rho : \mathcal{L} \to \mathrm{vect}(Y), \quad X_\mu \mapsto \sum_{a=1}^{r} X^a_\mu(y)\partial_{y^a},$$

determines a representation of $\mathcal{L}$, where $\mathrm{vect}(Y)$ denotes the infinite-dimensional Lie algebra of vector fields on Y.

Example 2.3 The prolongation algebra associated with the prolongation of Burgers' equation given by equations (2.15) is now identified. It is the free Lie algebra generated by $\{X_1, \ldots, X_4\}$ modulo the commutator relations

$$[X_1, X_2] = [X_1, X_3] = X_3, \quad [X_1, X_4] + [X_2, X_3] = 0, \quad [X_2, X_4] = 0.$$

Let $Y_0 = X_3$ and for each $n = 0, 1, \ldots$ let $Y_{n+1} = [X_2, Y_n]$. Since $[X_4, Y_0] = [X_4, [X_1, X_2]] = [[X_2, X_3], X_2] = -Y_2$ and

$$[X_4, Y_{n+1}] = [X_4, [X_2, Y_n]] = [X_2, [X_4, Y_n]],$$

it follows by induction on n that $[X_4, Y_n] = -Y_{n+2}$ for all $n = 0, 1, \ldots$. Suppose that $[Y_n, Y_{n+1}] = [Y_n, Y_{n+2}] = 0$ for some $n \in \{0, 1, \ldots\}$. Then

$$[Y_{n+1}, Y_{n+2}] = [Y_{n+1}, [Y_n, X_4]] = [Y_n, Y_{n+3}]$$

and

$$[Y_{n+1}, Y_{n+2}] = [[X_2, Y_n], Y_{n+2}] = -[Y_n, Y_{n+3}],$$

implying that $[Y_{n+1}, Y_{n+2}] = 0$. Further,

$$[Y_{n+1}, Y_{n+3}] = [Y_{n+1}, [X_2, Y_{n+2}]] = -[Y_{n+2}, Y_{n+2}] = 0.$$

Also, $[X_1, Y_1] = [X_1, [X_2, Y_0]] = [X_2, Y_0] = Y_1$, which implies that $[X_1, Y_2] = [X_1, [X_2, Y_1]] = [Y_0, Y_1] + Y_2$. However, $[X_1, Y_2] = [X_1, [Y_0, X_4]] = [Y_0, X_4] - [Y_0, Y_1] = Y_2 - [Y_0, Y_1]$ and it follows that $[Y_0, Y_1] = 0$. Then $[Y_0, Y_2] = [Y_0, [X_2, Y_1]] = 0$ and the induction step described above proves that $[Y_n, Y_{n+1}] = [Y_n, Y_{n+2}] = 0$ for all $n = 0, 1, \ldots$. Now suppose there exists a positive integer k such that for all nonnegative integers $j \leq k$

$$[Y_n, Y_{n+j}] = 0, \quad \forall n = 0, 1, \ldots .$$

Then for all nonnegative integers n,

$$[Y_n, Y_{n+k+1}] = [Y_n, [X_2, Y_{n+k}]] = -[Y_{n+1}, Y_{n+k}] + [X_2, [Y_n, Y_{n+k}]] = 0.$$

Notice that $[Y_n, Y_{n+1}] = 0$ for all nonnegative integers n, so that by induction on k, $[Y_m, Y_n] = 0$ for all $m, n = 0, 1, \ldots$. Finally, if $[X_1, Y_n] = Y_n$ then

$$[X_1, Y_{n+1}] = [X_1, [X_2, Y_n]] = [Y_0, Y_n] + [X_2, Y_n] = Y_{n+1}.$$

Since $[X_1, Y_0] = Y_0$, induction on n proves that $[X_1, Y_n] = Y_n$ for all $n = 0, 1, \ldots$.

At this stage, the following table of commutators has been obtained:

	X_1	X_2	X_4	Y_n
X_1	0	Y_0	$-Y_1$	Y_n
X_2		0	0	Y_{n+1}
X_4			0	$-Y_{n+2}$
Y_m				0

Since the commutator of every pair of elements introduced thus far is given, and because the Jacobi identity can be shown to hold in all cases, $\mathcal{L}$ must have basis $\{X_1, X_2, X_4, Y_n : n = 0, 1, \ldots\}$ and commutators as above. A simple relabelling presents this algebra in a much clearer form. Take as new basis $\{U_0, U_1, U_2, V_n : n = 0, 1, \ldots\}$, related to the old one by

$$U_0 = X_1, \quad U_1 = X_2 - Y_0, \quad U_2 = -X_4 - Y_1, \quad V_n = Y_n, \quad n = 0, 1, \ldots .$$

In terms of this basis the commutator table is

	U_c	V_d
U_a	0	V_{a+d}
V_b		0

where a, b, c and d take appropriate values. Thus, the prolongation algebra associated with equations (2.15) is a subalgebra of the loop algebra over the two-dimensional non-Abelian Lie algebra. This algebra is similar to one identified by Finley and McIver [24], although those authors have included an unnecessary central element in the algebra.

A representation of this algebra is given to demonstrate its role in obtaining explicit Wahlquist-Estabrook prolongations of Burgers' equation. The representation, which is in terms of vector fields on $\mathbb{R}$, is

$$U_m \mapsto \lambda^m y \partial_y, \quad V_m \mapsto 0,$$

where λ is an arbitrary constant. Therefore the dimension of the pseudopotential space is $r = 1$ and the corresponding solution to equations (2.14) is

$$X_1^1 = y, \quad X_2^1 = \lambda y, \quad X_3^1 = 0, \quad X_4^1 = -\lambda^2 y.$$

The prolongation equations

$$y_x = (u + \lambda) y, \quad y_t = (u_x + u^2 - \lambda^2) y,$$

describe a Wahlquist-Estabrook prolongation of Burgers' equation. □

Suppose that q of the prolongation equations $y_i^a = F_i^a$ can be written in the form $u^\alpha = G^\alpha(x, y^{(1)})$ for some smooth functions $\{G^\alpha : \alpha = 1, \ldots, q\}$. Using these expressions to eliminate u from (Δ, Ξ) yields a system of differential equations for $y(x)$, which will be called a *modified* version of Δ. This terminology is motivated by the famous "modified Korteweg-de Vries" equation, which can be derived from the Korteweg-de Vries equation in this manner (see [92] and Section 4.4).

Example 2.4 A modification of Burgers' equation can be constructed from the Wahlquist-Estabrook prolongation

$$y_x = (u + \lambda)y, \quad y_t = (u_x + u^2 - \lambda^2)y,$$

constructed in Example 2.3. From the equation for y_x, $u = y^{-1}y_x - \lambda$ and, after substituting this expression into the equation for y_t, one finds that $y(x,t)$ must satisfy

$$y_t = y_{xx} - 2\lambda y_x.$$

When $\lambda = 0$ this modification of Burgers' equation is the heat equation, so that the famous Hopf-Cole transformation [13], [44] has been derived by applying the Wahlquist-Estabrook method to Burgers' equation. □

Provided that one is prepared to accept a restricted class of solutions to equations (2.17) it is possible to replace the problem of constructing infinite-dimensional representations $\rho : \mathcal{L} \to \mathrm{vect}(Y)$ of the prolongation algebra with that of finding finite-dimensional representations. Equations (2.17) admit solutions of the form $X_\mu^a(y) = \sum_{b=1}^r (H_\mu)_b^a y^b$ if and only if the mapping

$$\rho : \mathcal{L} \to M_r(\mathbb{R}), \quad X_\mu \mapsto \mathbf{H}_\mu,$$

is a representation of $\mathcal{L}$. Here $\mathbf{H}_\mu$ is the matrix with constant entries $(H_\mu)_b^a$ and $M_r(\mathbb{R})$ is the Lie algebra of $r \times r$ matrices with real components and Lie bracket

$$[\mathbf{H}_\mu, \mathbf{H}_\nu] = \mathbf{H}_\nu \mathbf{H}_\mu - \mathbf{H}_\mu \mathbf{H}_\nu.$$

The resulting prolongation equations

$$y_i^a = \sum_{b=1}^r \left(\sum_{\mu=1}^s (H_\mu)_b^a \sigma_i^\mu \right) y^b, \quad i = 1, \ldots, p, \quad a = 1, \ldots, r,$$

are linear in the pseudopotentials and can be written in matrix form. If $\mathbf{y}$ is the vector-valued function of x with $\mathbf{y} = (y^1, \ldots, y^r)^T$ and $\mathbf{A}_i$ are the matrices with entries $(A_i)^a_b = \sum_{\mu=1}^{s}(H_\mu)^a_b \sigma^\mu_i$ then the prolongation equations can be written

$$\frac{\partial}{\partial x^i}\mathbf{y}(x) = \mathbf{A}_i\mathbf{y}, \quad i = 1, \ldots, p. \tag{2.18}$$

Equations (2.9) become

$$0 = D_{x^j}(\mathbf{A}_i) - D_{x^i}(\mathbf{A}_j) + \mathbf{A}_i\mathbf{A}_j - \mathbf{A}_j\mathbf{A}_i, \quad i, j = 1, \ldots, p, \tag{2.19}$$

which must be satisfied on $\mathcal{S}_\Delta$. Equations (2.19) determine a subvariety of $M^{(n)}$ in their own right. When this subvariety equals $\mathcal{S}_\Delta$ (it must always contain $\mathcal{S}_\Delta$), equations (2.19) are called a *zero-curvature representation* of Δ. Sometimes the related linear system of equations (2.18) will also be given this name. A zero-curvature representation of Δ

$$\frac{\partial}{\partial x^i}\mathbf{z}(x) = \mathbf{B}_i\mathbf{z}, \quad i = 1, \ldots, p, \tag{2.20}$$

is said to be *gauge-equivalent* to equations (2.18) if there exists a smooth $GL(r, \mathbb{R})$-valued function $\mathbf{G}$ on $M^{(n-1)}$ such that

$$\mathbf{B}_i = \mathbf{G}\mathbf{A}_i\mathbf{G}^{-1} + D_{x^i}(\mathbf{G})\mathbf{G}^{-1}, \quad i = 1, \ldots, p,$$

as then the equation $\mathbf{z} = \mathbf{G}\mathbf{y}$ relates solutions of equations (2.18) and (2.20). The mapping $\mathbf{y} \mapsto \mathbf{z} = \mathbf{G}\mathbf{y}$ is known as a *gauge transformation.*

If the prolongation equations for a Wahlquist-Estabrook prolongation are

$$y^a_i = F^a_i(x, u^{(n-1)}), \quad i = 1, \ldots, p, \quad a = 1, \ldots, r,$$

so that the right hand sides of these equations are independent of the pseudopotentials, then $y^1, \ldots, y^r$ are called *potentials.* When

$$0 = \Xi^a_i = y^a_i - F^a_i(x, u^{(n-1)}, y), \quad i = 1, \ldots, p, \quad a = 1, \ldots, r,$$

describe the prolongation and

$$\frac{\partial F^b_i}{\partial y^a} = 0, \quad i = 1, \ldots, p, \quad a = b, \ldots, r, \quad b = 1, \ldots, r,$$

it follows that (Δ, Ξ^1) is a prolongation of Δ which involves a single potential, y^1. Similarly, (Δ, Ξ^1, Ξ^2) is a prolongation of (Δ, Ξ^1) with potential y^2, and so on. In this case, $y^1, \ldots, y^r$ will be known as *nested potentials* and (Δ, Ξ) referred to as a *sequential prolongation* of Δ.

Consider an arbitrary Wahlquist-Estabrook prolongation of a differential equation Δ described by equations (2.10). For every smooth function $f : M^{(m)} \times Y \to \mathbb{R}$, the *prolonged total derivative of f with respect to x^i* is the function $\tilde{D}_{x^i} f : M^{(m+1)} \times Y \to \mathbb{R}$ defined by

$$\begin{aligned}
\tilde{D}_{x^i} f(x, u^{(m+1)}, y) &= \frac{\partial f}{\partial x^i} + \sum_{\alpha=1}^{q} \sum_{|J|=0}^{m} u^\alpha_{iJ} \frac{\partial f}{\partial u^\alpha_J} + \sum_{a=1}^{r} F^a_i \frac{\partial f}{\partial y^a} \\
&= D_{x^i} f + \sum_{a=1}^{r} F^a_i \frac{\partial f}{\partial y^a}.
\end{aligned}$$

Equations (2.9), which define a Wahlquist-Estabrook prolongation, can then be written

$$(\tilde{D}_{x^j}(F^a_i) - \tilde{D}_{x^i}(F^a_j))(x, u^{(n)}, y) = 0, \quad i, j = 1, \ldots, p, \quad a = 1, \ldots, r,$$

for all $(x, u^{(n)}) \in \mathcal{S}_\Delta$. The resulting prolongation (Δ, Ξ) is said to feature a *redundant pseudopotential* if there exists a smooth real-valued function θ on $M^{(n-1)} \times Y$ such that

1. $\tilde{D}_{x^i} \theta(x, u^{(n)}, y) = 0$ for all $i = 1, \ldots, p$, whenever $(x, u^{(n)}) \in \mathcal{S}_\Delta$, and
2. $\dfrac{\partial \theta}{\partial y^a}$ is nonzero on $\mathcal{S}_\Delta \times Y$ for some $a \in \{1, \ldots, r\}$.

A prolongation with no redundant pseudopotentials is called *nondegenerate*. Suppose that the pseudopotentials have been ordered in such a way that $a = r$ in condition 2. Then condition 1 indicates that θ is constant on solutions to Δ, say $\theta = K$, while condition 2 allows one to eliminate y^r from the prolongation equations entirely by solving $K = \theta(x, u^{(n-1)}, y)$ for y^r. This leaves a prolongation described by

$$y^b_i = F^b_i(x, u^{(n-1)}, y^1, \ldots, y^{r-1}, K), \quad i = 1, \ldots, p, \quad b = 1, \ldots, r-1,$$

which involves just $r - 1$ pseudopotentials as well as the constant parameter K.

For example, consider the prolongation of Burgers' equation defined by

$$y_x = (u + \lambda) y, \quad y_t = (u_x + u^2 - \lambda^2) y,$$
$$z_x = uz, \quad z_t = (u_x + u^2) z,$$

with two-dimensional pseudopotential space. One finds that

$$\tilde{D}_x\theta = \tilde{D}_t\theta = 0,$$

where $\theta = yz^{-1}\exp(-\lambda x + \lambda^2 t)$. $\dfrac{\partial\theta}{\partial y} \neq 0$ and one can eliminate y from the prolongation via $y = Kz\exp(\lambda x - \lambda^2 t)$, leaving the prolongation

$$z_x = uz, \; z_t = (u_x + u^2)z,$$

with one-dimensional pseudopotential space. Eliminating y has really not affected the system at all, since given $z(x,t)$ one can find $y(x,t)$ by putting $y(x,t) = Kz(x,t)\exp(\lambda x - \lambda^2 t)$, hence the description of y as a redundant pseudopotential.

This summary of Wahlquist-Estabrook prolongations concludes by briefly considering their symmetry structures. Due to the form of the prolongation equations, if Δ has maximal rank then (Δ, Ξ) must also enjoy this property. A similar result holds for local solvability — if Δ is locally solvable then so is (Δ, Ξ). Let $(x_0, u_0^{(n)}, y_0)$ be an arbitrary point in $\mathcal{S}_\Delta \times Y$ and, using the local solvability of Δ, let $\Phi : N \to M^{(n)}$ be a solution to Δ passing through $(x_0, u_0^{(n)})$. As mentioned above, the prolongation equations can be solved by foliating the submanifold $\tilde{\Phi} : N \times Y \to M^{(n)} \times Y$ using the prolongation forms $\{\omega^a : a = 1, \ldots, r\}$. The leaf containing $(x_0, u_0^{(n)}, y_0)$ yields the solution to (Δ, Ξ) which is needed to prove that the prolonged system is locally solvable. Consequently, when calculating symmetry groups of Wahlquist-Estabrook prolongations, one need only check that the original differential equation, Δ in this case, has maximal rank and is locally solvable.

The calculation of (classical) symmetry generators of (Δ, Ξ) can be simplified somewhat. In the usual approach, one would treat (Δ, Ξ) as a system of n-th order differential equations and require invariance of the subvariety $\mathcal{S}_{(\Delta,\Xi)}$ of $X \times (U \times Y)^{(n)}$. However, since the differential consequences of $\Xi[u, y] = 0$ contribute no extra equations to (Δ, Ξ), the latter system can be regarded as defining a subvariety of $M^{(n)} \times Y^{(1)} \subseteq X \times U^{(n)} \times Y^{(1)}$. Thus, all symmetry generators can be found by prolonging a vector field $\mathbf{v}$ on M to one on $M^{(n)} \times Y^{(1)}$ via

$$\mathrm{pr}^{(n)}\mathbf{v} = \sum_{i=1}^{p} \xi^i \partial_{x^i} + \sum_{\alpha=1}^{q} \sum_{|J|=0}^{n} \phi_J^\alpha \partial_{u_J^\alpha} + \sum_{a=1}^{r} \psi^a \partial_{y^a} + \sum_{i=1}^{p} \sum_{a=1}^{r} \psi_i^a \partial_{y_i^a},$$

and requiring that $\mathrm{pr}^{(n)}\mathbf{v}$ generates a local group of transformations leaving the subvariety $\mathcal{S}_{(\Delta,\Xi)}$ of $M^{(n)} \times Y^{(1)}$ invariant. In this process the functions appearing as coefficients in the coordinate expression for $\mathrm{pr}^{(n)}\mathbf{v}$ will be restricted to $\mathcal{S}_{(\Delta,\Xi)}$ — in particular, the substitution $y_i^a = F_i^a$ will be made — so that these coefficients can be evaluated using the prolonged total derivative operators introduced earlier. That is, one can assume that

$$\begin{aligned}
\phi_{Ji}^{\alpha} &= \tilde{D}_{x^i}\phi_J^{\alpha} - \sum_{j=1}^{p} u_{jJ}^{\alpha}\tilde{D}_{x^i}\xi^j, \\
\psi_i^a &= \tilde{D}_{x^i}\psi^a - \sum_{j=1}^{p} y_j^a \tilde{D}_{x^i}\xi^j \\
&= \tilde{D}_{x^i}\psi^a - \sum_{j=1}^{p} F_j^a \tilde{D}_{x^i}\xi^j,
\end{aligned}$$

when calculating symmetry generators. Then $\mathbf{v}$ is determined by the requirement that

$$\begin{aligned}
\mathrm{pr}^{(n)}\mathbf{v}(\Delta^l)(x, u^{(n)}, y) &= 0, \quad l = 1, \ldots, m, \\
\mathrm{pr}^{(n-1)}\mathbf{v}(\Xi_i^a)(x, u^{(n-1)}, y) &= 0, \quad i = 1, \ldots, p, \ \ a = 1, \ldots, r,
\end{aligned}$$

whenever $(x, u^{(n)}) \in \mathcal{S}_{\Delta}$.

Chapter 3

HC-projections

The foundation of this work is the generalization of the Hopf-Cole transformation which is developed in the current chapter. Section 3.1 reviews the history of the transformation and its underlying geometry is studied in the following section. Geometric "extended problems" are defined in Section 3.2 and then a method is described which converts these into systems of differential equations. Section 3.3 exploits symmetry groups of differential equations, together with the extended problems of the preceding section, to construct generalizations of the Hopf-Cole transformation. New equations related to the heat equation are discovered. When applied to ordinary differential equations this method reduces to Lie's technique for reducing the order of such equations, a topic dealt with in Section 3.4. Ovsjannikov's "partially-invariant" solutions and the "side conditions" of Olver and Rosenau are also discussed there. They provide other potential applications for extended problems. Finally, Section 3.5 uses the generalized Hopf-Cole transformations to construct auto-Bäcklund transformations for differential equations. A considerably more powerful method for constructing auto-Bäcklund transformations will be described in Chapter 5.

3.1 A brief history of the Hopf-Cole transformation

This chapter is motivated by two differential equations and the relationship between them. If $v(x, t)$ is a solution of the heat equation $v_t = v_{xx}$ then $u(x, t) = v^{-1}v_x$ will be a solution of Burgers' equation $u_t = u_{xx} + 2uu_x$, introduced by Burgers [6] as a model for turbulence. Hopf [44] and Cole [13] are jointly credited with the discovery of the transformation $v \mapsto u$, which is now known as the *Hopf-Cole transformation*, although it was certainly known to Forsyth (see [32], Chap. 21, Vol. 6) as early as 1906. This transformation allows one to associate a particular solution of Burgers' equation with each solution of the heat equation. Conversely, given a solution $u(x,t)$ of Burgers' equation, by solving the system of first order equations

$$v_x = uv, \quad v_t = (u_x + u^2)v,$$

for $v(x,t)$, one obtains a family of solutions to the heat equation parametrized by a single constant of integration.

Since the discovery, or rediscovery, of the Hopf-Cole transformation there have been many attempts at generalization. A popular method is to make small changes to the transformation or the linear equation, or perhaps both, and examine the new nonlinear equations which emerge. Chu [12], King [53] and Sachdev [79] adopt this approach. Whitham [96] has applied a variation of the Hopf-Cole transformation to the Korteweg-de Vries (KdV) equation, using the resulting nonlinear equation to discuss the interaction of solitary waves. Kumei and Bluman [57] give necessary and sufficient conditions for a system of nonlinear differential equations to be related to a linear system by an (invertible) contact transformation. The Hopf-Cole transformation is, of course, noninvertible and so is not an example of the mappings considered by Kumei and Bluman. However, their work does arise in another approach to the Hopf-Cole transformation. Following Whitham [96], one replaces Burgers' equation by its potential version

$$w_t = w_{xx} + w_x^2,$$

where $w(x,t)$ is defined by $w(x,t) = \int^x u(x',t)dx'$. This equation admits the symmetry generator $\theta(x,t)e^{-w}\partial_w$, where $\theta_t = \theta_{xx}$, with the results of Kumei and Bluman immediately yielding the transformation $w(x,t) = \log v(x,t)$ relating the heat and potential Burgers' equations and, from it, recovering the Hopf-Cole transformation. Recently, this approach has received much attention as it is an example of constructing a *nonlocal symmetry generator* of a differential equation [5], [52]. The material presented in Section 5.2 greatly eases the search for such nonlocal symmetries.

This chapter presents a generalization of the Hopf-Cole transformation which is obtained by systematically exploiting the symmetry structure of a differential equation. With each symmetry group of a differential equation this method associates a new equation related to the original one by a generalized Hopf-Cole transformation. The new equation is analogous to Burgers' equation and the original one to the heat equation. After this and the following chapter had been prepared [36], a closely related paper by Sokolov, Svinolupov and Wolf [87] appeared in the literature. A detailed comparison of the two methods appears in Appendix A where this author will argue that the technique outlined in the present chapter is both easier to implement and potentially more widely applicable.

3.2 The r-extended problem

A close examination of the Hopf-Cole transformation motivates the generalization which will be introduced soon. It is instructive to study the way in which a solution of Burgers' equation leads to a one-parameter family of solutions to the heat equation. Suppose that $u(x,t)$ satisfies Burgers' equation. Then, as stated above, one must solve

$$0 = v_x - uv, \quad 0 = v_t - (u_x + u^2)v, \tag{3.1}$$

for $v(x,t)$ in order to obtain the corresponding solutions of the heat equation. Instead of solving these differential equations, interpret them as algebraic equations defining a three-dimensional submanifold of the jet space $M^{(1)}$. Here $M = X \times U$ with $X = \mathbb{R}^2$ and $U = \mathbb{R}^1$ having coordinates (x,t) and v respectively. Thus $M^{(1)} = \mathbb{R}^5$ has coordinates (x,t,v,v_x,v_t). When dealing with solutions of the heat equation one is interested in submanifolds of $M^{(2)} = \mathbb{R}^8$, which has coordinates $(x,t,v,v_x,v_t,v_{xx},v_{xt},v_{tt})$. The submanifold of $M^{(1)}$ described by equations (3.1) prolongs to a submanifold of $M^{(2)}$ by adding the differential consequences of these equations to obtain a larger system. In this case, the maximal set of independent equations

$$\begin{aligned}
0 &= v_x - uv, \\
0 &= v_t - (u_x + u^2)v, \\
0 &= v_{xx} - (u_x + u^2)v, \\
0 &= v_{xt} - (u_{xx} + 3uu_x + u^3)v,
\end{aligned}$$

$$0 = v_{tt} - (u_{xxx} + 4uu_{xx} + 3u_x^2 + 6u^2u_x + u^4)v,$$

describes the appropriate submanifold $\Phi : N \to M^{(2)}$. This submanifold is three-dimensional and contained in $\mathcal{S}_\Delta$, the subvariety of $M^{(2)}$ corresponding to the heat equation, since $v_t - v_{xx} = 0$ on $\Phi(N)$. Also, when restricted to $\Phi(N)$, the contact module $\Omega^{(2)}$ is spanned by a single one-form

$$\omega = dv - uvdx - (u_x + u^2)vdt$$

and generates an ideal of forms which is closed under exterior differentiation. Thus, $\Phi(N)$ admits a foliation of dimension $\dim \Phi(N) - \dim \Omega^{(2)}|_{\Phi(N)} = 2$, each leaf corresponding to a solution of the heat equation.

This example has uncovered a geometric problem which can be posed for any system of differential equations. As well as looking for particular solutions to the system, one can search for submanifolds of the appropriate jet space which foliate into solutions of the differential equation. This problem is now formally defined.

Definition 3.1 Let $\Delta[u] = 0$ denote a system of n-th order differential equations defined on $M \subseteq X \times U$, where $X = \mathbb{R}^p$ and $U = \mathbb{R}^q$. For each nonnegative integer $r \leq q(p+n-1)!/(p!(n-1)!)$ a solution to the *r-extended problem associated with* Δ is any $(p+r)$-dimensional submanifold $\Phi : N \to M^{(n)}$ of $M^{(n)}$ such that the following conditions hold:

1. $\Phi(N) \subseteq \mathcal{S}_\Delta$,
2. $\Phi^*\Omega^{(n)}$ is spanned by r independent one-forms and
3. $\Phi^*\Omega^{(n)}$ generates an ideal of forms closed under exterior differentiation. □

The integer r cannot take values greater than $q(p+n-1)!/(p!(n-1)!)$ in Definition 3.1 due to the fact that the contact module $\Omega^{(n)}$ on $M^{(n)}$ is generated by only that number of independent one-forms. Fortunately, this upper bound can easily be avoided. If one wishes to construct an r-extended problem for larger values of r it is sufficient to consider not the differential equation Δ, but the higher order system comprising Δ and all differential consequences of suitably high order. By increasing the order n of the system in this manner, r is effectively unbounded above.

It has been shown that each solution of Burgers' equation yields a solution to the one-extended problem associated with the heat equation. Each such solution then leads to a one-parameter family of solutions of the heat equation itself. This result is generalized in the following proposition, which proves that each solution to the r-extended problem associated with a differential equation yields a family of solutions to that equation which involves r parameters.

Proposition 3.2 *Let $\Delta[u] = 0$ denote a system of n-th order differential equations on $M \subseteq X \times U$, where $X = \mathbb{R}^p$ and $U = \mathbb{R}^q$, and suppose that $\Phi : N \to M^{(n)}$ is a solution to the r-extended problem associated with Δ. Then N admits a codimension r foliation $\{N_\gamma : \gamma \in \Gamma\}$ such that each submanifold $\Phi : N_\gamma \to M^{(n)}$ is a solution to Δ. Furthermore, if $\Psi : P \to M^{(n)}$ is a solution of Δ such that $\Psi(P) \subset \Phi(N)$ then there exists $\gamma \in \Gamma$ such that $\Psi(P) \subseteq \Phi(N_\gamma)$.*

PROOF: Let $\Phi : N \to M^{(n)}$ be a solution to the r-extended problem associated with Δ. Then $\Phi^*\Omega^{(n)}$ is spanned by r independent one-forms and generates an ideal of forms which is closed under exterior differentiation. It therefore determines a codimension r foliation $\{N_\gamma : \gamma \in \Gamma\}$ of N. Notice that

$$\Phi(N_\gamma) \subseteq \Phi(N) \subseteq \mathcal{S}_\Delta$$

and $\Phi^*\Omega^{(n)} = 0$ when restricted to N_γ. Therefore each, necessarily p-dimensional, submanifold $\Phi : N_\gamma \to M^{(n)}$ must be a solution of Δ.

The second part follows from the fact that $\Phi^{-1} \circ \Psi(P)$ is a submanifold of N on which $\Phi^*\Omega^{(n)} = 0$. Recall that the foliation of N is determined by the differential ideal generated by $\Phi^*\Omega^{(n)}$, so that $\Phi^{-1} \circ \Psi(P)$ must be contained within some leaf of the foliation. The result now follows immediately. □

The geometric formulation of the r-extended problem, as featured in Definition 3.1, is very useful when analyzing the theory of transformations generalizing the one found by Hopf and Cole, but is extremely cumbersome to work with in applications. Fortunately, if Δ is a system of differential equations involving p independent variables it is possible to represent the r-extended problem by a system of differential equations involving $p + r$ independent variables.

Let $\Phi : N \to M^{(n)}$ be a solution of the r-extended problem associated with an n-th order differential equation $\Delta[u] = 0$ defined on $M \subseteq X \times U$, where $X = \mathbb{R}^p$

and $U = \mathbb{R}^q$ have coordinates $x = (x^1, \dots, x^p)$ and $u = (u^1, \dots, u^q)$ respectively. Choose local coordinates $(w, z) = (w^1, \dots, w^r, z^1, \dots, z^p)$ for N such that $\Phi^*\Omega^{(n)}$ is generated by the one-forms $\{\xi^a : a = 1, \dots, r\}$, where

$$\xi^a = dw^a + \sum_{i=1}^{p} F_i^a(w, z) dz^i, \quad a = 1, \dots, r,$$

for suitable smooth functions F_i^a. Property 2 of Definition 3.1 implies that the pullback of each contact form is a linear combination of the forms $\{\xi^a : a = 1, \dots, r\}$. Since $\Omega^{(n)}$ is spanned by $\{\theta_I^\alpha : \alpha = 1, \dots, q,\ 0 \le |I| \le n-1\}$, where

$$\theta_I^\alpha = du_I^\alpha - \sum_{j=1}^{p} u_{Ij}^\alpha dx^j, \quad \alpha = 1, \dots, q, \quad 0 \le |I| \le n-1,$$

there must exist smooth functions G_{Ia}^α on N such that

$$\begin{aligned} 0 &= \Phi^*\theta_I^\alpha + \sum_{a=1}^{r} G_{Ia}^\alpha \xi^a \\ &= \sum_{a=1}^{r} \left(G_{Ia}^\alpha + \frac{\partial u_I^\alpha}{\partial w^a} - \sum_{j=1}^{p} u_{Ij}^\alpha \frac{\partial x^j}{\partial w^a} \right) dw^a \\ &\quad + \sum_{i=1}^{p} \left(\sum_{a=1}^{r} G_{Ia}^\alpha F_i^a + \frac{\partial u_I^\alpha}{\partial z^i} - \sum_{j=1}^{p} u_{Ij}^\alpha \frac{\partial x^j}{\partial z^i} \right) dz^i, \\ &\qquad \alpha = 1, \dots, q, \quad 0 \le |I| \le n-1. \end{aligned}$$

Eliminating G_{Ia}^α from these equations yields the overdetermined linear system of algebraic equations for $\{F_i^a : a = 1, \dots, r,\ i = 1, \dots, p\}$,

$$\begin{gathered} \sum_{a=1}^{r} \left(\frac{\partial u_I^\alpha}{\partial w^a} - \sum_{j=1}^{p} u_{Ij}^\alpha \frac{\partial x^j}{\partial w^a} \right) F_i^a = \frac{\partial u_I^\alpha}{\partial z^i} - \sum_{j=1}^{p} u_{Ij}^\alpha \frac{\partial x^j}{\partial z^i}, \\ i = 1, \dots, p, \quad \alpha = 1, \dots, q, \quad 0 \le |I| \le n-1. \end{gathered} \tag{3.2}$$

The consistency conditions for this system lead to a system of first order partial differential equations for the components of Φ — that is, for $x = x(w, z)$ and so on. Once these consistency conditions have been satisfied, the functions F_i^a can be found in terms of the components of Φ and their derivatives. A further set of differential equations for these components comes from the closure condition

$$0 = d\xi^a \wedge \xi^1 \wedge \cdots \wedge \xi^r, \quad a = 1, \dots, r.$$

Finally, there is the system of algebraic equations for the components of Φ which follows from the condition $\Phi(N) \subseteq \mathcal{S}_\Delta$.

In summary, the r-extended problem can be reformulated as the system of differential equations derived from

1. the consistency conditions for the linear system of equations (3.2),

2. the closure condition and

3. the requirement that $\Phi(N) \subseteq \mathcal{S}_\Delta$.

This system will be called an *r-extended equation associated with* Δ.

The final form which an r-extended equation takes will depend on the way in which the submanifold $\Phi : N \to M^{(n)}$ has been described. Different choices of parametrization for $\Phi(N)$ will lead to extended equations which, although describing the same geometric problem, can differ radically in appearance. For this reason, studying generalized Hopf-Cole transformations on the basis of extended *equations* would be hopelessly coordinate dependent. Instead, the generalization will be based on the, coordinate independent, extended *problem.* In particular, theoretical results will be given in terms of extended problems. To transfer this theory into results concerning the projected equations introduced in Section 3.3, one must first specify a parametrization of solutions to the relevant extended problem.

Some choices of parametrization lead to unnecessarily complicated extended equations, even for relatively simple examples like the heat equation. The best way to parametrize $\Phi : N \to M^{(n)}$ is often to choose $p + r$ coordinates on $M^{(n)}$ as parameters (that is, as the independent variables for the r-extended equation) and let all other coordinates depend on them. These other coordinates will then be the dependent variables of the r-extended equation. The process of calculating extended equations is demonstrated by the following example.

Example 3.3 This example constructs a one-extended equation associated with the heat equation $v_t = v_{xx}$. Here $X = \mathbb{R}^2$ and $U = \mathbb{R}^1$ have coordinates (x, t) and v, respectively. $M = X \times U$ and, since the heat equation is second order, the appropriate jet space is $M^{(2)}$, with coordinates $(x, t, v, v_x, v_t, v_{xx}, v_{xt}, v_{tt})$. The contact forms are

$$\theta = dv - v_x dx - v_t dt,$$

$$\begin{aligned}\theta_x &= dv_x - v_{xx}dx - v_{xt}dt,\\ \theta_t &= dv_t - v_{xt}dx - v_{tt}dt,\end{aligned}$$

and span $\Omega^{(2)}$. The heat equation is described by the subvariety $\mathcal{S}_\Delta = \ker\Delta$ of $M^{(2)}$ where

$$\Delta : M^{(2)} \to \mathbb{R}^1, \quad \Delta = v_t - v_{xx}.$$

The one-extended problem associated with the heat equation involves finding all three-dimensional submanifolds of $\mathcal{S}_\Delta$ such that, on them, $\Omega^{(2)}$ is spanned by a single one-form and generates a closed ideal of differential forms. The coordinates (x, t, v) of M will be used to parametrize these submanifolds, which will be described by

$$v_x = w(x,t,v), \quad v_t = m(x,t,v),$$

and similarly for the second order derivative terms. Taking Φ as the inclusion mapping, the one-form

$$\eta = \Phi^*\theta = dv - w(x,t,v)dx - m(x,t,v)dt$$

is everywhere nonzero and thus serves as a generator for $\Phi^*\Omega^{(2)}$. From the coefficient of dv,

$$\Phi^*\theta_x = dw(x,t,v) - v_{xx}dx - v_{xt}dt = w_v\eta,$$

and, equating coefficients of dx and dt, one obtains

$$v_{xx} = w_x + ww_v, \quad v_{xt} = w_t + w_v m. \tag{3.3}$$

Similarly, $\Phi^*\theta_t = m_v\eta$ and

$$v_{xt} = m_x + wm_v, \quad v_{tt} = m_t + mm_v. \tag{3.4}$$

The closure condition $0 = d\eta \wedge \eta$ yields the equation

$$0 = w_t - m_x + w_v m - wm_v$$

which is already contained in the combined system of equations (3.3) and (3.4), so is not needed here. All that remains is the subvariety condition $v_t = v_{xx}$ which becomes

$$m = w_x + ww_v. \tag{3.5}$$

Thus. taking (x, t, v) as coordinates on the sought-after submanifold, the one-extended problem reduces to solving the system of first order differential equations comprising equations (3.3) to (3.5). Eliminating m leads to a single, second order, partial differential equation

$$w_t = w_{xx} + 2ww_{xv} + w^2 w_{vv}$$

which, from now on, will be referred to as the *first extension of the heat equation.* It is, of course, just one of many possible one-extended equations associated with the heat equation. However, all such equations represent a common geometric problem.

Given a solution of this extended equation, the one-form

$$\eta = dv - wdx - mdt = dv - wdx - (w_x + ww_v)dt$$

determines the foliation resulting in solutions of the heat equation. □

It is possible to construct extended equations without using differential forms at all — even the geometric extended problem can be avoided. Having chosen coordinates on $M^{(n)}$ to act as independent variables for the r-extended equation, the r-dimensionality of $\Phi^*\Omega^{(n)}$ amounts to the chain rule. In the example above, since $v_x = w(x, t, v)$, the subvariety condition and then the chain rule imply that

$$v_t = v_{xx} = \frac{\partial w}{\partial x} + \frac{\partial w}{\partial v} \cdot \frac{\partial v}{\partial x} = w_x + ww_v.$$

Closure of the contact module reflects the commutativity of mixed derivatives. For the heat equation, this yields the equation

$$\begin{aligned} 0 &= \frac{\partial}{\partial t}(v_x) - \frac{\partial}{\partial x}(v_t) \\ &= (w_t + w_v(w_x + ww_v)) - (w_{xx} + w_x w_v + ww_{vx} + w(w_{xv} + w_v^2 + ww_{vv})) \\ 0 &= w_t - w_{xx} - 2ww_{xv} - w^2 w_{vv}, \end{aligned}$$

recovering the first extension of the heat equation. This technique is very useful computationally and extended equations for many systems of differential equations are easily constructed in this manner. However, as will become clear later, the geometric nature of the underlying extended problems has many uses.

3.3 HC-projected problems

It was seen in the preceding section that each solution $u(x,t)$ of Burgers' equation yields a solution to the one-extended problem associated with the heat equation. To motivate the generalization of the Hopf-Cole transformation it is sufficient to identify what makes these solutions of the one-extended problem special. The distinguishing feature can be expressed in three different ways. Let $G = \{g_a : a \in \mathbb{R}\}$ denote the symmetry group of the heat equation defined by

$$g_a : M \to M, \quad (x,t,v) \mapsto (x,t,e^a v), \quad a \in \mathbb{R}.$$

Then the solution $\Phi : N \to M^{(2)}$ of the one-extended problem determined by $u(x,t)$ is invariant under the action of $\mathrm{pr}^{(2)}G$ on $M^{(2)}$. An alternative formulation of this feature involves the first extension of the heat equation. Each solution $u(x,t)$ of Burgers' equation yields a solution $w(x,t,v) = v \cdot u(x,t)$ of the first extension of the heat equation — a solution which must be invariant under the symmetry group $\tilde{G} = \{\tilde{g}_a : a \in \mathbb{R}\}$ of this equation defined by

$$\tilde{g}_a : (x,t,v,w) \mapsto (x,t,e^a v, e^a w), \quad a \in \mathbb{R}.$$

Finally, notice that Burgers' equation arises as the $\tilde{G}$-reduction of the first extension of the heat equation.

The groups G and $\tilde{G}$ are intimately related. Essentially, the symmetry group G of the heat equation is extended to obtain the symmetry group $\tilde{G}$ of the first extension of the heat equation. Such behaviour is not unique to this symmetry group of this equation. The following proposition shows that each symmetry group of any differential equation prolongs to yield a symmetry group of each associated r-extended problem. Furthermore, if the resulting symmetry group leaves a solution of the extended problem invariant, it will also preserve the foliation of that solution into solutions of the original differential equation. In the case of the heat equation, this means that the members of the one-parameter family of solutions to the heat equation derived from a particular solution of Burgers' equation are mapped into one another under the action of $\mathrm{pr}^{(2)}G$.

Proposition 3.4 *Let G be a symmetry group of a system $\Delta[u] = 0$ of n-th order differential equations on $M \subseteq X \times U$. Suppose that $\Phi : N \to M^{(n)}$ is a solution of the r-extended problem associated with Δ.*

1. *For all* $g \in G$ *for which* $\mathrm{pr}^{(n)}g \cdot (\Phi(N))$ *is defined,* $\mathrm{pr}^{(n)}g \cdot \Phi : N \to M^{(n)}$ *is also a solution of the* r*-extended problem associated with* Δ.

2. *Let* $\{N_\gamma : \gamma \in \Gamma\}$ *be the foliation of* N *such that each submanifold* $\Phi : N_\gamma \to M^{(n)}$ *is a solution of* Δ. *If* $\Phi(N)$ *is* $\mathrm{pr}^{(n)}G$*-invariant, then to each* $g \in G$ *and* $\gamma \in \Gamma$ *there corresponds* $g(\gamma) \in \Gamma$ *such that*

$$\mathrm{pr}^{(n)}g \cdot \Phi(N_\gamma) \subseteq \Phi(N_{g(\gamma)}).$$

Proof: Given the foliation $\{N_\gamma : \gamma \in \Gamma\}$ of N such that each $\Phi : N_\gamma \to M^{(n)}$ is a solution to Δ, and the fact that $g \in G$ is a symmetry of Δ, it follows that $\mathrm{pr}^{(n)}g \cdot \Phi : N_\gamma \to M^{(n)}$ is a solution of Δ for all $\gamma \in \Gamma$. Consequently,

$$\mathrm{pr}^{(n)}g \cdot \Phi(N) = \mathrm{pr}^{(n)}g \cdot \Phi\left(\bigcup_{\gamma\in\Gamma} N_\gamma\right) \subseteq \mathcal{S}_\Delta.$$

Also, if the ideal of forms generated by $\Phi^*\Omega^{(n)}$ is closed under exterior differentiation and generated by r independent one-forms, then the same is clearly true for

$$\left(\mathrm{pr}^{(n)}g \cdot \Phi\right)^* \Omega^{(n)} = \Phi^*\left((\mathrm{pr}^{(n)}g)^*\Omega^{(n)}\right) = \Phi^*\Omega^{(n)}.$$

That is, $\mathrm{pr}^{(n)}g \cdot \Phi : N \to M^{(n)}$ is a solution of the r-extended problem associated with Δ, proving the first part of the proposition.

To prove the second part, recall from above that $\mathrm{pr}^{(n)}g \cdot \Phi : N_\gamma \to M^{(n)}$ is a solution of Δ and, since

$$\mathrm{pr}^{(n)}g \cdot \Phi(N_\gamma) \subseteq \mathrm{pr}^{(n)}g \cdot \Phi(N) = \Phi(N),$$

Proposition 3.2 implies that

$$\mathrm{pr}^{(n)}g \cdot \Phi(N_\gamma) \subseteq \Phi(N_{g(\gamma)})$$

for some $g(\gamma) \in \Gamma$. □

Given a differential equation Δ it is now possible to associate another differential equation with each symmetry group G of Δ. After choosing a parametrization for its solutions, one simply constructs the $(\dim G)$-extended equation associated with Δ and finds the symmetry group $\tilde{G}$ of this new equation which corresponds to G. The

$\tilde{G}$-reduction of the extended equation will, provided the action of $\tilde{G}$ has $(\dim G)$-dimensional orbits, involve the same number of independent variables as Δ. It is the analogue of Burgers' equation, related to Δ by a generalization of the Hopf-Cole transformation. The precise manner in which solutions of the two equations are related will be discussed after the following definition, which describes these generalized Hopf-Cole transformations in coordinate-independent terms.

Definition 3.5 Let G be an r-dimensional symmetry group of a system $\Delta[u] = 0$ of n-th order differential equations on $M \subseteq X \times U$, where r is bounded as in Definition 3.1. Suppose that every $\mathrm{pr}^{(n)}G$-orbit is r-dimensional. The submanifold $\Phi : N \to M^{(n)}$ is called a solution of the *G-induced HC-projected problem associated with* Δ if it is a locally $\mathrm{pr}^{(n)}G$-invariant solution of the r-extended problem associated with Δ. The *order* of this HC-projection equals r and $\Pi_G(\Delta)$ is used to denote the G-induced HC-projected problem associated with Δ. □

The $\tilde{G}$-reduction of the $(\dim G)$-extended equation associated with Δ, which was described immediately prior to Definition 3.5, will be called a *G-induced HC-projected equation associated with* Δ.

Solutions to Δ are closely related to those of $\Pi_G(\Delta)$ for any symmetry group G of Δ. By Proposition 3.2, every solution to $\Pi_G(\Delta)$, because it is a solution to the $(\dim G)$-extended problem associated with Δ, must foliate into a $(\dim G)$-parameter family of solutions to Δ. Conversely, consider the submanifold of the jet space $M^{(n)}$ which corresponds to some solution of Δ. If the differential equation involves p independent variables and G is r-dimensional, then the collection of $\mathrm{pr}^{(n)}G$-orbits through this submanifold is a submanifold of $M^{(n)}$ with dimension less than or equal to $p + r$. The action of $\mathrm{pr}^{(n)}G$ on this submanifold clearly defines a foliation into solutions of Δ. One would like to be able to say that when this submanifold has dimension equal to $p+r$ it must be a solution to the r-extended problem associated with Δ, and hence a solution to $\Pi_G(\Delta)$. However, it is necessary to prove that the restriction of the contact module to this submanifold is spanned by r independent one-forms before such a claim can be made. The following technical lemma is useful.

Lemma 3.6 *Let $M \subseteq X \times U$ be an open subset, with $X = \mathbb{R}^p$ and $U = \mathbb{R}^q$, and suppose that $\Phi : N \to M^{(n)}$ is a $(p + r)$-dimensional submanifold with $0 \leq r \leq$*

$q(p+n-1)!/(p!(n-1)!)$. *If* $\pi^n_{n-1} \circ \Phi : N \to M^{(n-1)}$ *is a* $(p+r)$*-dimensional submanifold of* $M^{(n-1)}$ *then* $\Phi^*\Omega^{(n)}$ *is spanned by at least* r *independent one-forms.*
PROOF: Let X, U and N have coordinates $x = (x^1, \ldots, x^p)$, $u = (u^1, \ldots, u^q)$ and $y = (y^1, \ldots, y^{p+r})$ respectively, and suppose that

$$\theta = \sum_{\alpha=1}^{q} \sum_{|I|=0}^{n-1} f^I_\alpha(x, u^{(n)}) \theta^\alpha_I$$

is an element of the contact module. Then $\Phi^*\theta = 0$ if and only if

$$\sum_{\alpha=1}^{q} \sum_{|I|=0}^{n-1} f^I_\alpha \frac{\partial u^\alpha_I}{\partial y^a} - \sum_{i=1}^{p} \left(\sum_{\alpha=1}^{q} \sum_{|I|=0}^{n-1} f^I_\alpha u^\alpha_{Ii} \right) \frac{\partial x^i}{\partial y^a} = 0, \quad a = 1, \ldots, p+r. \tag{3.6}$$

Let $s = q(p+n-1)!/(p!(n-1)!)$ so that $\Omega^{(n)}$ is spanned by s one-forms. If $\Phi^*\Omega^{(n)}$ is spanned by k independent one-forms then there must be $s-k$ independent equations of the form of equations (3.6). Consequently, the matrix

$$\mathsf{J}(y) = \begin{pmatrix} \dfrac{\partial x^i}{\partial y^a} \\ \dfrac{\partial u^\alpha_I}{\partial y^a} \end{pmatrix},$$

where $|I|$ ranges over $\{0, \ldots, n-1\}$, must have rank less than or equal to $p+k$ since it has $p+s$ rows and $s-k$ dependence relations given by equations (3.6). The fact that $\pi^n_{n-1} \circ \Phi : N \to M^{(n-1)}$ is a $(p+r)$-dimensional submanifold and that $\mathsf{J}(y)$ is the corresponding Jacobian matrix implies that $p + r = \operatorname{rank}(\mathsf{J}(y)) \le p + k$. Therefore, $r \le k$ and $\Phi^*\Omega^{(n)}$ is spanned by at least r independent one-forms. □

Unfortunately, the rather awkward restriction in Lemma 3.6 that $\pi^n_{n-1} \circ \Phi : N \to M^{(n-1)}$ must be $(p+r)$-dimensional is necessary. Suppose that $X = \mathbb{R}^1$, $U = \mathbb{R}^2$ and $N = \mathbb{R}^3$ have coordinates t, (u, v) and x, y, z respectively. The three-dimensional submanifold of $M^{(1)}$ described by

$$t = x, \quad u = y, \quad v = x^2, \quad u_t = z, \quad v_t = 2x,$$

projects onto a two-dimensional submanifold of $M^{(0)}$. Here $n = p = 1$ and $q = r = 2$ and the pullback of the contact module is spanned by just $\{dy - z dx\}$, not the two independent one-forms one would have hoped for.

Let $\Phi : N \to M^{(n)}$ be a solution to the differential equation Δ described on $M \subseteq X \times U$, with $X = \mathbb{R}^p$. Suppose that G is an r-dimensional symmetry group of

Δ, with r bounded as in Lemma 3.6, such that the submanifold $\pi^n_{n-1}(\mathrm{pr}^{(n)}G\cdot\Phi(N))$ of $M^{(n-1)}$ is $(p+r)$-dimensional. By Lemma 3.6, the restriction of $\Omega^{(n)}$ to $\mathrm{pr}^{(n)}G\cdot\Phi(N)$ is spanned by at least r independent one-forms. Since the ideal of forms determining the foliation of $\mathrm{pr}^{(n)}G\cdot\Phi(N)$ into solutions of Δ must contain at most r independent one-forms it follows that this foliation is determined solely by the restriction of $\Omega^{(n)}$, which therefore generates an ideal closed under exterior differentiation. As a consequence of this, $\mathrm{pr}^{(n)}G\cdot\Phi(N)$ is a solution to the r-extended problem associated with Δ. The manifest $\mathrm{pr}^{(n)}G$-invariance of this submanifold proves that it is also a solution to $\Pi_G(\Delta)$. This proves the following result.

Proposition 3.7 *Let G be an r-dimensional symmetry group of a system $\Delta[u]=0$ of n-th order differential equations on $M\subseteq X\times U$ where $X=\mathbb{R}^p$, $U=\mathbb{R}^q$ and r is bounded as in Definition* 3.1.

1. *If $\Phi: N\longrightarrow M^{(n)}$ is a solution to $\Pi_G(\Delta)$ then there exists a foliation $\{N_\gamma:\gamma\in\Gamma\}$ of N such that each $\Phi: N_\gamma\longrightarrow M^{(n)}$ is a solution to Δ. The action of $\mathrm{pr}^{(n)}G$ preserves the resulting foliation $\{\Phi(N_\gamma):\gamma\in\Gamma\}$ of $\Phi(N)$.*

2. *If $\Phi: N\longrightarrow M^{(n)}$ is a solution of Δ such that $\mathrm{pr}^{(n-1)}G\cdot(\pi^n_{n-1}\circ\Phi(N))$ is $(p+r)$-dimensional then $\mathrm{pr}^{(n)}G\cdot\Phi(N)$ is a solution to $\Pi_G(\Delta)$.* □

The restriction that $\mathrm{pr}^{(n-1)}G\cdot(\pi^n_{n-1}\circ\Phi(N))$ be $(p+r)$-dimensional can be relaxed somewhat. Suppose that $\mathrm{pr}^{(n)}G\cdot\Phi(N)$ is $(p+r)$-dimensional but that its projection onto $M^{(n-1)}$ is not. By prolonging the solution $\Phi: N\longrightarrow M^{(n)}$ of Δ to the $(n+1)$-th order jet space one obtains a submanifold $\mathrm{pr}^{(1)}\Phi: N\longrightarrow M^{(n+1)}$. Because $\pi^{n+1}_n(\mathrm{pr}^{(n+1)}G\cdot\mathrm{pr}^{(1)}\Phi(N))=\mathrm{pr}^{(n)}G\cdot\Phi(N)$ is $(p+r)$-dimensional, it follows that $\mathrm{pr}^{(n+1)}G\cdot\mathrm{pr}^{(1)}\Phi(N)$ satisfies the requirements of Lemma 3.6, indicating that the restriction of the contact module $\Omega^{(n+1)}$ to $\mathrm{pr}^{(1)}\Phi$ must be spanned by at least r independent one-forms. By repeating the arguments before Proposition 3.7 one finds that this submanifold is a solution to the G-induced HC-projected problem associated with Δ, described this time on the $(n+1)$-th order jet space.

As with HC-projected problems, the solutions to HC-projected equations are closely related to solutions of the original differential equation Δ. A solution to some HC-projected equation immediately yields a solution to the corresponding extended problem, which can then be foliated into a multi-parameter family of solutions to

Δ. This process will often be described as *lifting* solutions from the HC-projected equation up to solutions of Δ. Conversely, one can *project* solutions of Δ onto solutions of an HC-projected equation. Subject to the technical considerations discussed above, a solution to Δ yields a $\mathrm{pr}^{(n)}G$-invariant solution to the $(\dim G)$-extended problem associated with Δ. This will be equivalent to a $\tilde{G}$-invariant solution of some $(\dim G)$-extended equation, leading immediately to a solution of the corresponding G-induced HC-projected equation.

When written in terms of local coordinates, the relationship between solutions of a differential equation and solutions of an HC-projected equation can be expressed in a form similar to the Hopf-Cole transformation. This is demonstrated in the following example, where all (in a sense to be described later) HC-projections of order one of the heat equation are constructed.

Example 3.8 The one-extended problem associated with the heat equation was constructed in Example 3.3 and led to the differential equation

$$w_t = w_{xx} + 2ww_{xv} + w^2 w_{vv} \tag{3.7}$$

when solutions of the one-extended problem were described by $v_x = w(x, t, v)$. Each symmetry generator of the heat equation prolongs to one of equation (3.7) as follows:

$$\begin{aligned}
\mathbf{v}_1 &= \partial_x, \\
\mathbf{v}_2 &= \partial_t, \\
\mathbf{v}_3 &= v\partial_v + w\partial_w, \\
\mathbf{v}_4 &= x\partial_x + 2t\partial_t - w\partial_w, \\
\mathbf{v}_5 &= 2t\partial_x - xv\partial_v - (xw + v)\partial_w, \\
\mathbf{v}_6 &= 4xt\partial_x + 4t^2\partial_t - (x^2 + 2t)v\partial_v - (x^2 w + 2xv + 6tw)\partial_w, \\
\mathbf{v}_\theta &= \theta(x, t)\partial_v + \theta_x(x, t)\partial_w.
\end{aligned}$$

$\theta(x, t)$ is an arbitrary solution of the heat equation $\theta_t = \theta_{xx}$. Notice that by restricting these vector fields to (x, t, v)-space the symmetry algebra of the heat equation is recovered [72]. Reducing equation (3.7) using a one-dimensional symmetry group generated by an element of the Lie algebra spanned by the vectors above yields a differential equation which is a first order HC-projection of the heat equation. An "optimal system" of one-dimensional subalgebras of this Lie algebra is provided by those subalgebras spanned by

(a) $a\mathbf{v}_3 + \mathbf{v}_4$,
(b) $\mathbf{v}_2 + b\mathbf{v}_3 + \mathbf{v}_6$,
(c) $\mathbf{v}_2 - \mathbf{v}_5$,
(d) $\mathbf{v}_2 + c\mathbf{v}_3$,
(e) $\mathbf{v}_1$,
(f) $\mathbf{v}_3$ and
(g) $\mathbf{v}_\theta$,

where $a, b, c \in \mathbb{R}$ and $\theta_t = \theta_{xx}$ [72]. The term optimal system will be defined after this example, which constructs the HC-projections of the heat equation corresponding to each of these subalgebras. The first case will be treated in detail, the others more briefly.

(a) A maximal, functionally independent, set of invariants of the symmetry group generated by $a\mathbf{v}_3 + \mathbf{v}_4$ is $\{t^{-1/2}x,\ t^{-a/2}v,\ t^{-(a-1)/2}w\}$. In order to find solutions of the first extension of the heat equation invariant under this group, one therefore makes the *Ansatz* $w(x,t,v) = t^{(a-1)/2}u(y,z)$ where $y = t^{-1/2}x$ and $z = t^{-a/2}v$. Equation (3.7) then reduces to

$$0 = 2(u_{yy} + 2uu_{yz} + u^2u_{zz}) + yu_y + u + a(zu_z - u) \tag{3.8}$$

which, by the preceding theory, is related to the heat equation by an HC-projection of order one. If $v(x,t)$ is a solution of the heat equation then the corresponding solution of equation (3.8) is given parametrically by defining y, z and u as the following functions of (x,t):

$$y = t^{-1/2}x, \quad z = t^{-a/2}v(x,t), \quad u = t^{(1-a)/2}v_x(x,t). \tag{3.9}$$

Treating these equations as defining a mapping from $\mathbb{R}^2$, with coordinates (x,t), into $\mathbb{R}^3$, with coordinates (y,z,u), one finds that the corresponding Jacobian matrix is

$$\begin{pmatrix} t^{-1/2} & -\frac{1}{2}t^{-3/2}x \\ t^{-a/2}v_x & t^{-a/2}(v_t - \frac{a}{2}t^{-1}v) \\ t^{(1-a)/2}v_{xx} & t^{(1-a)/2}(v_{xt} - \frac{a-1}{2}t^{-1}v_x) \end{pmatrix},$$

which has rank one if and only if

$$xv_x + 2tv_t - av = 0.$$

This is precisely the condition that $v(x,t)$ be invariant under the symmetry group generated by $x\partial_x + 2t\partial_t + av\partial_v$, the symmetry generator of the heat equation which corresponds to $a\mathbf{v}_3 + \mathbf{v}_4$. All but such invariant solutions of the heat equation yield solutions to equation (3.8). Eliminating x and t between equations (3.9) leads to solutions $u = u(y,z)$ in standard form. Conversely, to lift a solution $u(y,z)$ of equation (3.8) up to a family of solutions of the heat equation, one must solve the first order system

$$v_x = w = t^{(a-1)/2}u, \quad v_t = w_x + ww_v = t^{(a-2)/2}(u_y + uu_z),$$

for $v(x,t)$, after making the substitution $y = t^{-1/2}x$ and $z = t^{-a/2}v(x,t)$. Such systems will often be very difficult to solve, mainly because the parametrization of solutions to the heat equation using (x,t) is unnatural here. A better description of solutions would involve writing x, t and v directly as functions of y and z. This approach will be pursued in Example 4.1.

(b) The symmetry group generated by $\mathbf{v}_2 + b\mathbf{v}_3 + \mathbf{v}_6$ leads to invariant solutions of the first extension of the heat equation of the form

$$w(x,t,v) = \frac{u(y,z) - 2tyz}{(1+4t^2)^{3/4}} \exp\left(\frac{b}{2}\arctan(2t) - y^2 t\right),$$

where

$$y = \frac{x}{(1+4t^2)^{1/2}} \tag{3.10}$$

and

$$z = \frac{v\,(1+4t^2)^{1/4}}{\exp\left(\dfrac{b}{2}\arctan(2t) - \dfrac{x^2 t}{1+4t^2}\right)}. \tag{3.11}$$

The reduced equation is then

$$0 = u_{yy} + 2uu_{yz} + u^2u_{zz} + (zu_z - u)(b - y^2) + 2yz, \tag{3.12}$$

which is another first order HC-projection of the heat equation. Mapping of solutions from the heat equation onto solutions of equation (3.12) is determined by equations (3.10) and (3.11), together with

$$u = \frac{\left(v_x + \frac{2xtv}{1+4t^2}\right)(1+4t^2)^{3/4}}{\exp\left(\frac{b}{2}\arctan(2t) - \frac{x^2 t}{1+4t^2}\right)}.$$

Conversely, solutions to the heat equation corresponding to a solution $u(y, z)$ of the projected equation are found by solving the first order system

$$\begin{aligned} v_x &= \frac{u - 2tyz}{(1+4t^2)^{3/4}} \exp\left(\frac{b}{2}\arctan(2t) - y^2 t\right), \\ v_t &= \frac{(u_y + uu_z - 4tyu - 2tz + 4t^2y^2z)}{(1+4t^2)^{5/4}} \exp\left(\frac{b}{2}\arctan(2t) - y^2 t\right), \end{aligned}$$

for $v(x, t)$, after making the substitutions given by equations (3.10) and (3.11).

(c) A solution of the first extension of the heat equation will be invariant under the symmetry group generated by $\mathbf{v}_2 - \mathbf{v}_5$ if it has the form

$$w(x, t, v) = (u(y, z) + tz)\exp\left(ty - \frac{t^3}{3}\right),$$

where $y = x + t^2$ and $z = v \cdot \exp(-xt - 2t^3/3)$, and if the reduced equation

$$0 = u_{yy} + 2uu_{yz} + u^2 u_{zz} + yzu_z - yu - z$$

is satisfied. The equations defining y and z, together with

$$u = (v_x - tv)\exp\left(-xt - \frac{2t^3}{3}\right),$$

describe how solutions of the heat equation yield solutions of the HC-projected equation. To map solutions in the other direction, one must solve the first order equations

$$\begin{aligned} v_x &= (u + tz)\exp\left(ty - \frac{t^3}{3}\right), \\ v_t &= (u_y + uu_z + 2tu + t^2 z)\exp\left(ty - \frac{t^3}{3}\right), \end{aligned}$$

for $v(x, t)$.

(d) The infinitesimal symmetry $\mathbf{v}_2+c\mathbf{v}_3$ of the first extension of the heat equation generates group-invariant solutions of the form $w(x,t,v) = u(x,y)\cdot\exp(ct)$, where $y = v\cdot\exp(-ct)$ and $u(x,y)$ must satisfy the reduced equation

$$0 = u_{xx} + 2uu_{xy} + u^2u_{yy} + c(yu_y - u).$$

Each solution of the heat equation immediately leads to a solution of this HC-projected equation, via $u = v_x\cdot\exp(-ct)$ and the equation defining y. If $u(x,y)$ is a solution of the HC-projected equation, then the corresponding solutions of the heat equation satisfy

$$v_x = ue^{ct}, \quad v_t = (u_x + uu_y)e^{ct}.$$

(e) Solutions of the first extension of the heat equation invariant under the group generated by $\mathbf{v}_1$ take the form $w(x,t,v) = u(t,v)$, where $u(t,v)$ must satisfy the reduced equation

$$u_t = u^2u_{vv}.$$

Given a solution of the heat equation, the corresponding solution of the HC-projected equation is described parametrically by $u = v_x$. Conversely, if $u(t,v)$ satisfies the HC-projected equation then solving the first order system

$$v_x = u, \quad v_t = uu_v,$$

leads to a one-parameter family of solutions to the heat equation. This relationship has been noted already by Rosen [78].

(f) If a solution of the first extension of the heat equation is to be invariant under the group generated by $\mathbf{v}_3$ it must be of the form $w(x,t,v) = v\cdot u(x,t)$. Here $u(x,t)$ satisfies the reduced equation

$$u_t = u_{xx} + 2uu_x$$

which is Burgers' equation. Solutions are mapped between the heat and Burgers' equation as has been described previously.

(g) A solution of the first extension of the heat equation invariant under the group generated by $\mathbf{v}_\theta$ has the form $w(x,t,v) = \theta^{-1}\theta_x v + u(x,t)$ and the resulting reduced equation is

$$u_t = u_{xx} + 2\theta^{-2}(\theta\theta_{xx} - \theta_x^2)u.$$

Solutions are mapped from the heat equation to the HC-projected equation by

$$u = v_x - \theta^{-1}\theta_x v.$$

Conversely, each solution of the HC-projected equation leads to the one-parameter family of solutions to the heat equation which arise as solutions to

$$\begin{aligned} 0 &= v_x - \theta^{-1}\theta_x v - u, \\ 0 &= v_t - \theta^{-1}\theta_{xx} v - u_x - \theta^{-1}\theta_x u. \end{aligned}$$

Table 3.1 displays these first order HC-projections of the heat equation. The first column gives the projected equation and the projections are described in column two. Only expressions for any new variables introduced are given there. The first order equations used in lifting solutions of the projected equations up to solutions of the heat equation are not included, as a more efficient method for performing this process will be described in Example 4.1. □

Example 3.8 claims to give all first order HC-projections of the heat equation. Recall that if a differential equation is reduced using conjugate subgroups of its full symmetry group, then there exists a symmetry of that differential equation which transforms each solution invariant under one of the subgroups into a solution invariant under the other subgroup. This symmetry is precisely the group element through which the two subgroups are conjugate. Suppose the differential equation occurs on M and has symmetry group G. If H and K are subgroups of G with $K = gHg^{-1}$ for some $g \in G$, then the H- and K-reduced equations, appearing on M/H and M/K respectively, are related by the diffeomorphism

$$\tilde{g} : M/H \to M/K, \quad H \cdot x \mapsto K \cdot (g \cdot x),$$

where $H \cdot x$ denotes the H-orbit through $x \in M$ and $K \cdot (g \cdot x)$ the K-orbit through $g \cdot x$. A similar situation occurs for HC-projected problems. If two conjugate subgroups induce HC-projected problems then those two problems will be equivalent via some coordinate change. Therefore, to classify up to diffeomorphism all HC-projections of order r of a differential equation one must classify all r-dimensional subgroups of the symmetry group of that differential equation up to conjugation. Equivalently, assuming all symmetry groups are connected, one must classify all r-dimensional subalgebras of the symmetry algebra of that differential equation up to

	Projected Equation	HC-Projection
(a)	$0 = 2(u_{yy} + 2uu_{yz} + u^2u_{zz})$ $\quad + yu_y + u + a(zu_z - u)$	$y = t^{-1/2}x$ $z = t^{-a/2}v(x,t)$ $u = t^{(1-a)/2}v_x(x,t)$
(b)	$0 = u_{yy} + 2uu_{yz} + u^2u_{zz}$ $\quad + (zu_z - u)(b - y^2) + 2yz$	$y = \dfrac{x}{(1+4t^2)^{1/2}}$ $z = \dfrac{v\,(1+4t^2)^{1/4}}{\exp\left(\frac{b}{2}\arctan(2t) - \frac{x^2t}{1+4t^2}\right)}$ $u = \dfrac{\left(v_x + \frac{2xtv}{1+4t^2}\right)(1+4t^2)^{3/4}}{\exp\left(\frac{b}{2}\arctan(2t) - \frac{x^2t}{1+4t^2}\right)}$
(c)	$0 = u_{yy} + 2uu_{yz} + u^2u_{zz}$ $\quad + yzu_z - yu - z$	$y = x + t^2$ $z = v \cdot \exp(-xt - 2t^3/3)$ $u = (v_x - tv)\exp(-xt - 2t^3/3)$
(d)	$0 = u_{xx} + 2uu_{xy} + u^2u_{yy}$ $\quad + c(yu_y - u)$	$y = v \cdot \exp(-ct)$ $u = v_x \cdot \exp(-ct)$
(e)	$u_t = u^2u_{vv}$	$u = v_x$
(f)	$u_t = u_{xx} + 2uu_x$	$u = v^{-1}v_x$
(g)	$u_t = u_{xx} + 2\theta^{-2}(\theta\theta_{xx} - \theta_x^2)u$	$u = v_x - \theta^{-1}\theta_x v$

Table 3.1: First order HC-projections of the heat equation

conjugation. A set of such subalgebras comprises an *optimal system* of r-dimensional subalgebras [72].

A final comment should be made about Example 3.8. The system of one-dimensional subalgebras given there is not, strictly speaking, optimal since, for many solutions $\alpha(x,t)$ and $\beta(x,t)$ of the heat equation, the subalgebras generated by $\mathbf{v}_\alpha$ and $\mathbf{v}_\beta$ will be conjugate. However, it is more efficient to allow all solutions θ, and accept some redundancy, than to attempt a classification of the subalgebras generated by $\mathbf{v}_\theta$.

Just as the previous section concluded with an outline of efficient ways to construct extended equations avoiding the use of extended problems, so this section ends by demonstrating a technique for constructing HC-projected equations. The approach followed here avoids not only the HC-projected problem, but also any appearance of the extended equation. Given a symmetry group G of a system of n-th order differential equations $\Delta[u] = 0$ described on $M \subseteq X \times U$, with $X = \mathbb{R}^p$, one begins by constructing a maximal, functionally independent set of invariants of the $\mathrm{pr}^{(n)}G$-action on $M^{(n)}$. After choosing p of these invariants to act as independent variables for the projected system, one writes the other invariants as unknown functions of the former ones. The next step involves selecting $\dim G$ parametric variables in such a way that each coordinate on $M^{(n)}$ can be written as a function of these parametric variables and the p special invariants. If the parametric variables and the p distinguished invariants only involve terms on $M^{(k)}$, for some $k < n$, then the coordinates of $M^{(n)}$ can be obtained from those of $M^{(k+1)}$ via the chain rule. The projected system of equations then arises from the subvariety condition representing Δ, together with integrability conditions which appear while calculating higher order derivatives.

Example 3.9 The HC-projection of the heat equation corresponding to case (a) in Example 3.8 is rederived using the technique just outlined. The symmetry group G of the heat equation with infinitesimal generator $x\partial_x + 2t\partial_t + av\partial_v$ yields a prolonged group action on $M^{(2)}$ with invariants

$$t^{-1/2}x, \quad t^{-a/2}v, \quad t^{-(a-1)/2}v_x, \quad t^{-(a-2)/2}v_t,$$
$$t^{-(a-2)/2}v_{xx}, \quad t^{-(a-3)/2}v_{xt}, \quad t^{-(a-4)/2}v_{tt}.$$

Invariants $y = t^{-1/2}x$ and $z = t^{-a/2}v$ are chosen as independent variables for the

HC-projected system, so that

$$t^{-(a-1)/2}v_x = u(y,z), \quad t^{-(a-2)/2}v_t = s(y,z),$$

for unknown functions u and s, and similarly for the other invariants. Taking t as parametric variable, it follows that $k = 0$ for this example. Since

$$v_x = t^{(a-1)/2}u(y,z), \quad v_t = t^{(a-2)/2}s(y,z),$$

it follows that v_{xx}, v_{xt} and v_{tt} can be found by differentiation, as claimed. Only v_{xx} and v_{xt} are needed here, with

$$v_{xx} = t^{(a-2)/2}(u_y + uu_z)$$

and v_{xt} having two expressions. From v_x it follows that

$$v_{xt} = -\tfrac{1}{2}t^{(a-3)/2}(yu_y + (az - 2s)u_z + (1-a)u),$$

while v_t leads to

$$v_{tx} = t^{(a-3)/2}(s_y + us_z).$$

The subvariety condition forces $s = u_y + uu_z$ and the integrability condition $v_{xt} = v_{tx}$ means that

$$\begin{aligned} 0 &= 2s_y + 2us_z + yu_y + (az - 2s)u_z + (1-a)u \\ &= 2(u_{yy} + 2uu_{yz} + u^2u_{zz}) + yu_y + u + a(zu_z - u). \end{aligned}$$

Thus, equation (3.8) has been recovered using only the symmetry generator $x\partial_x + 2t\partial_t + av\partial_v$ of the heat equation. □

3.4 Extended problems and reduction techniques

Integration of ordinary differential equations

Most of the remainder of this study will be devoted to applications of transformations relating partial differential equations. Now, though, the behaviour of HC-projections when applied to ordinary differential equations will be discussed. It will be demonstrated that the construction of HC-projected equations associated with ordinary differential equations coincides with Lie's method for integrating such equations.

The simplest case of a single ordinary differential equation with one dependent variable is considered. Let the differential equation Δ be

$$0 = \Delta\left(x, u, u_1, \ldots, u_n\right),$$

where solutions are of the form $u = u(x)$ and

$$u_k = \frac{d^k u}{d\,x^k}, \quad k = 1, \ldots, n.$$

Suppose that G is an r-dimensional symmetry group of Δ. In order to construct an r-extended equation associated with Δ, describe solutions of the r-extended problem by

$$u_r = v\left(x, u, u_1, \ldots, u_{r-1}\right),$$

for a function v of $r+1$ variables. Using the procedure described at the end of Section 3.2 to construct the r-extended equation, successive application of the chain rule implies that

$$u_{r+k} = D^k(v)$$

where D is the differential operator

$$D = \frac{\partial}{\partial x} + u_1 \frac{\partial}{\partial u} + \cdots + u_{r-1} \frac{\partial}{\partial u_{r-2}} + v \frac{\partial}{\partial u_{r-1}}.$$

Thus, u_{r+k} involves k-th order partial derivatives of the dependent variable v of the extended equation. Since no integrability conditions need to be considered, the resulting r-extended equation associated with Δ is an $(n-r)$-th order partial differential equation with one dependent and $r+1$ independent variables. This equation admits an r-dimensional symmetry group $\tilde{G}$ derived from the symmetry group G of Δ. The corresponding reduction of the extended equation will be a single $(n-r)$-th order ordinary differential equation for one dependent variable provided that all $\tilde{G}$-orbits are r-dimensional. As with Lie's method, the r-dimensional symmetry group G enables one to reduce the order of Δ by r [72]. Systems of ordinary differential equations can be treated similarly.

One could describe the concept of HC-projections as a generalization of Lie's method from ordinary differential equations to partial differential equations. However, with ordinary differential equations these methods yield lower order projected equations which actually help in the construction of the general solution to the original system. With a partial differential equation, the projected equations are not necessarily any simpler to solve than the original one. For instance, Example 3.8 constructed, up to coordinate changes, all first order HC-projected equations associated with the heat equation. Like that equation, these equations all involved one dependent and two independent variables and were all second order differential equations. In fact, almost all of the projected equations were nonlinear and so much more difficult to solve than the heat equation itself! Thus, for differential equations involving $p > 1$ independent variables, HC-projections will not necessarily assist in the construction of general solutions, although, as will be shown in the following section, they can help one construct particular solutions.

Partially-invariant solutions

There exists a very successful method for constructing particular solutions to partial differential equations which involves searching for solutions invariant under a particular symmetry group of that differential equation. Such solutions are described by a system of equations involving fewer independent variables than the original differential equation — usually r less variables, where r equals the dimension of the symmetry group being used. This technique was described in Section 2.3. Ovsjannikov [75] generalized this construction by describing a method which yielded so-called *partially-invariant solutions* involving, once more, a system of reduced equations with fewer independent variables. Extended problems, as defined in Section 3.2, can be used to study these partially-invariant solutions. The analysis presented here does not pretend to be exhaustive, but is intended to indicate the potential to use extended problems to study these interesting, but much neglected, special solutions.

Ovsjannikov's generalization of group-invariant solutions involves searching for solutions $\Phi : N \to M^{(n)}$ of $\Delta[u] = 0$, an n-th order differential equation defined on $M \subseteq X \times U$, which satisfy the following requirement. If $X = \mathbb{R}^p$ and G, a symmetry group of Δ, is such that all $\mathrm{pr}^{(n)}G$-orbits are r-dimensional then $\mathrm{pr}^{(n)}G \cdot \Phi(N)$, the collection of $\mathrm{pr}^{(n)}G$-orbits through $\Phi(N)$, must have dimension strictly less than $p+r$. The generalization arises from the possibility that this dimension is greater than p — clearly, when it equals p, $\Phi : N \to M^{(n)}$ is a genuinely G-invariant solution of Δ. Suppose that $\mathrm{pr}^{(n)}G \cdot \Phi(N)$ is $(p+s)$-dimensional, with $0 \leq s < r$. Then, because $\mathrm{pr}^{(n)}g \cdot \Phi : N \to M^{(n)}$ is clearly a solution to Δ for each $g \in G$ such that $\mathrm{pr}^{(n)}g \cdot \Phi(N)$

is defined, it follows that $\mathrm{pr}^{(n)}G\cdot\Phi(N)$ admits a codimension s foliation into solutions of Δ. By Lemma 3.6, if $\pi^n_{n-1}(\mathrm{pr}^{(n)}G\cdot\Phi(N))$ is $(p+s)$-dimensional then $\mathrm{pr}^{(n)}G\cdot\Phi(N)$ is a solution to the s-extended problem associated with Δ. It is obviously $\mathrm{pr}^{(n)}G$-invariant. Therefore, to construct solutions of Δ which are partially-invariant under the action of G, for each integer s satisfying $1-p+r \leq s < r$, construct an s-extended equation associated with Δ. In each case there exists a symmetry group $\tilde{G}_s$ of this equation corresponding to the symmetry group G of Δ. The partially-invariant solutions are then determined by the $\tilde{G}_s$-reduction of the s-extended equation and this reduced equation must then involve just $p-r+s$ independent variables. Noticing that $1 \leq p-r+s < p$, it follows that this equation involves fewer independent variables than Δ.

Example 3.10 The heat equation $v_t = v_{xx}$ admits a two-dimensional symmetry group G with infinitesimal generators ∂_t and $v\partial_v$. In this case, $M = \mathbb{R}^2 \times \mathbb{R}^1$ has coordinates (x,t,v) and, on the open subset where $v \neq 0$, the action of G has two-dimensional orbits, whence $r = 2$. Since $p = 2$ in this example, the bounds on s are $1 \leq s < 2$, so that partially-invariant solutions arise from reducing any one-extended equation associated with the heat equation. Such an equation has already been calculated in Example 3.3:

$$w_t = w_{xx} + 2ww_{xv} + w^2 w_{vv} \tag{3.13}$$

G leads to a symmetry group $\tilde{G}_1$ of equation (3.13) with infinitesimal generators ∂_t and $v\partial_v + w\partial_w$. Thus, $\tilde{G}_1$-invariant solutions take the form $w(x,t,v) = v\cdot f(x)$ for some function f. Upon substituting this *Ansatz* into equation (3.13) one obtains the reduced ordinary differential equation

$$0 = \ddot{f} + 2f\dot{f}.$$

Integrating once, one obtains $\dot{f} + f^2 = a$ for some constant a, which has general solution

$$f(x) = \begin{cases} -\lambda\tan(\lambda x+\delta), & a = -\lambda^2,\ \lambda > 0, \\ \lambda\tanh(\lambda x+\delta), & a = \lambda^2,\ \lambda > 0, \\ (x+\delta)^{-1}, & a = 0. \end{cases}$$

The corresponding partially-invariant solutions to the heat equation are found by solving

$$v_x = w = v\cdot f(x), \quad v_t = w_x + ww_v = v(\dot{f}+f^2) = av.$$

This process corresponds to foliating a solution to the one-extended problem. The solutions are

$$v(x,t) = \begin{cases} ke^{-\lambda^2 t}\cos(\lambda x + \delta), & a = -\lambda^2, \lambda > 0, \\ ke^{\lambda^2 t}\cosh(\lambda x + \delta), & a = \lambda^2, \lambda > 0, \\ k(x+\delta), & a = 0, \end{cases}$$

which, interestingly, arise as genuinely invariant solutions under the symmetry group generated by $\partial_t + av\partial_v$ (see Example 3.17 of [72]). □

As Example 3.10 has shown, construction of partially-invariant solutions, at least as far as determining the "partially-reduced" equations, is straightforward once an appropriate extended equation has been calculated. It is believed that the interpretation of such solutions using the notion of extended problems may be able to significantly advance the understanding of this process. In particular, extended problems and HC-projections may be able to be used to answer such questions as which symmetry groups of a differential equation yield partially-invariant solutions which are not totally invariant under some other symmetry group of that differential equation. Such a study is beyond the scope of this section.

In the process of revising this chapter, the author discovered papers by Sastri and Dunn [82] and Sastri, Dunn and Rao [83] purporting to apply Ovsjannikov's method to the heat equation. They seem to misinterpret his ideas, since they seek solutions partially-invariant under one-dimensional symmetry groups. From the restriction $0 \leq s < r$ with $r = \dim G = 1$, it follows that only $s = 0$ yields partially-invariant solutions, and that then they must be totally invariant under G. What Sastri *et al.* have done is seek solutions of the heat equation such that G maps the corresponding submanifold of the jet space into a manifold of dimension less than or equal to $p + r$, rather than of dimension strictly less than $p + r$. Clearly, these authors are not considering a restricted set of solutions, as the term partial-invariance implies, but rather the entire set of solutions to the heat equation. In doing this, they have constructed what are called HC-projected equations here, since any solution $\Phi : N \to M^{(n)}$ of Δ such that $\mathrm{pr}^{(n)}G \cdot \Phi : N \to M^{(n)}$ is $(p+r)$-dimensional, with r equalling the dimension of the orbits of the $\mathrm{pr}^{(n)}G$-action, leads to a solution of the G-induced HC-projected problem associated with Δ, provided that $\pi^n_{n-1}(\mathrm{pr}^{(n)}G \cdot \Phi(N))$ is also $(p+r)$-dimensional.

It must be stressed that HC-projections, as presented in this chapter, were found independently by the current author. Furthermore, Sastri *et al.* did not seem to realize that the equations they constructed determined the entire class of solutions to the heat equation, rather than just invariant or partially-invariant solutions. As a result, the relationship with the Hopf-Cole transformation was not observed.

Extended problems as side conditions

Olver and Rosenau [73] suggest that almost all constructions of particular solutions to partial differential equations involve appending certain *side conditions* to the equations and solving the resulting overdetermined system. They provide a unifying framework for such diverse techniques as calculating group-invariant solutions and partially-invariant solutions as well as solving differential equations using separation of variables. Of all the possible side conditions which one may append to a differential equation, those having the following properties are of special interest:

1. *Compatibility*, meaning that the combined system of equations has solutions.

2. *Solubility*, meaning that the combined system is easier to solve than the original one.

As pointed out by Olver and Rosenau, "what is now required is an algorithmic method of determining these compatible side conditions." Evidence will be presented here which suggests that the extended problems defined in Section 3.2 are a useful tool when analyzing side conditions and may provide such an algorithmic method. As with the discussion concerning partially-invariant solutions, the aim here is merely to indicate possible further applications for extended problems.

Each solution to an extended problem associated with a particular differential equation leads to a system of compatible side conditions, obtained by writing the submanifold of the appropriate jet space implicitly. The side conditions are then the equations determining this submanifold. Compatibility is assured by Proposition 3.2, which proved that each system of side conditions derived in this manner leads to a multi-parameter family of solutions to the differential equation being studied. Conversely, many (but not all) compatible side conditions can be interpreted as solutions to an extended problem associated with the differential equation.

Consider the example of the heat equation $v_t = v_{xx}$ and the side condition

$$\phi(x,t,v) - \xi(x,t,v)v_x - \tau(x,t,v)v_t = 0, \tag{3.14}$$

which reflects invariance of solutions under the group generated by

$$\mathbf{v} = \xi(x,t,v)\partial_x + \tau(x,t,v)\partial_t + \phi(x,t,v)\partial_v.$$

When $\tau \neq 0$, equation (3.14) can be rewritten as

$$v_t = f(x,t,v)v_x + g(x,t,v). \tag{3.15}$$

Taking total x- and t-derivatives of this equation and using the heat equation one finds that

$$\begin{aligned} v_{xx} &= fv_x + g, \\ v_{xt} &= (D_x f + f^2)v_x + (D_x g + fg), \\ v_{tt} &= (D_t f + fD_x f + f^3)v_x + (D_t g + fD_x g + f^2 g). \end{aligned} \tag{3.16}$$

Equations (3.15) and (3.16) describe a four-dimensional subvariety of the second order jet space. It can be shown using the methods of Section 3.2 that this is actually a solution to the two-extended problem associated with the heat equation if and only if

$$f(x,t,v) = -\alpha(x,t), \quad g(x,t,v) = \beta(x,t) + v \cdot \gamma(x,t),$$

where the unknown functions α, β and γ satisfy

$$\begin{aligned} 0 &= \alpha_t - \alpha_{xx} + 2\alpha\alpha_x + 2\gamma_x, \\ 0 &= \beta_t - \beta_{xx} + 2\alpha_x\beta, \\ 0 &= \gamma_t - \gamma_{xx} + 2\alpha_x\gamma. \end{aligned}$$

These are exactly the conditions obtained by Bluman and Cole [3] for $\mathbf{v}$ to yield a nonclassical symmetry reduction of the heat equation. Thus, for the case of the heat equation, at least, one can construct conditional symmetry groups by seeking solutions to the two-extended problem of the form given by equation (3.15). The other case to consider has $\tau = 0$. Assuming that $\xi \neq 0$, one can rewrite equation (3.14) as

$$v_x = w(x,t,v)$$

and. repeating the steps above, obtain a three-dimensional subvariety of the second order jet space. From Example 3.3, this is a solution to the one-extended problem associated with the heat equation if and only if w satisfies the first extension of the heat equation. Once more, this is exactly the condition required for $\mathbf{v}$ to generate a conditional symmetry group.

This example suggests a close relationship between conditional symmetry groups and extended problems. Preliminary investigations suggest that the determining equations for the infinitesimal generators of such groups can be obtained by seeking solutions of appropriate extended equations which satisfy certain linearity assumptions. For a scalar n-th order differential equation involving p independent variables, one makes the *Ansatz*

$$u_{x^{i_1}} = \sum_{j=2}^{p} u_{x^{i_j}} \xi^j(x,u) + \phi(x,u), \tag{3.17}$$

where $\{i_1, \ldots, i_p\} = \{1, \ldots, p\}$, for solutions to an associated n-extended equation. Each solution appears to yield a conditional symmetry group generated by

$$\partial_{x^{i_1}} - \sum_{j=2}^{p} \xi^j(x,u)\partial_{x^{i_j}} + \phi(x,u)\partial_u.$$

Since any solution to an extended problem yields compatible side conditions, it may be fruitful to consider various *Ansätze*, rather than just the linear ones described by equation (3.17). Suppose, for example, that solutions to the three-extended problem associated with the Korteweg-de Vries (KdV) equation $0 = u_t + u_{xxx} + 12uu_x$ are sought of the form

$$u_t = cu_x^{-1}u_{xx}^2 + A(u, u_x),$$

with c constant and A a smooth function of the indicated arguments. One finds that the only possible solution of this form with $c \neq 0$ has $c = -\frac{1}{2}$ and

$$A(u, u_x) = \frac{-4uu_x^2 + 20u^4 + 2k_1(u_x^2 + 4u^3) + 2k_2 + 2k_3u + 2k_4u^2}{2u_x}.$$

The side condition is thus

$$0 = -u_t + \frac{-u_{xx}^2 - 4uu_x^2 + 20u^4 + 2k_1(u_x^2 + 4u^3) + 2k_2 + 2k_3u + 2k_4u^2}{2u_x}$$

and can be converted into an equation involving only u and its x-derivatives by replacing u_t with $-u_{xxx} - 12uu_x$. Among the differential consequences of this side condition is the equation

$$0 = u_{xxxxx} + 20uu_{xxx} + 40u_xu_{xx} + 120u^2u_x + 2k_1(u_{xxx} + 12uu_x) + 2k_4u_x,$$

used by Lax [60] to derive the two-soliton solution for the KdV equation. Recall that, unlike the one-soliton solution of the KdV equation, the two-soliton solution cannot be obtained via the nonclassical symmetry reduction method. That is, the linearity assumption of equation (3.17) must be relaxed before the two-soliton solution can be obtained.

Attention need not be restricted to n-extended equations. For instance, the side conditions corresponding to additive and multiplicative separation of variables for the heat equation which featured in [73] both describe solutions to the three-extended problem associated with the heat equation.

Finally, Olver and Rosenau provide a group-theoretic interpretation of their side conditions using generalized symmetries, but admit that this interpretation is "somewhat artificial." While the approach involving extended problems does not pretend to offer any group-theoretic insight, it does, at least, place many of the side conditions considered by those authors in a more geometric setting.

3.5 Auto-Bäcklund transformations

A major application for HC-projections is as an interpretive tool, used *a posteriori*. The relationship between HC-projections and Wahlquist-Estabrook prolongations is developed in Section 4.1. A group-theoretic interpretation of the interrelationships between various integrable equations which arise very naturally while prolonging the KdV equation is given in Section 4.5. When armed with HC-projections as well as another new notion, to be introduced in Section 5.2, the Wahlquist-Estabrook prolongation method becomes even more powerful. Many problems resistant to attack by the prolongation method alone, succumb to this combined arsenal. Such an enhanced approach to the problem of constructing Bäcklund transformations is the subject of Section 5.5. However, HC-projections have applications independent of the technique of Wahlquist and Estabrook. For example, some auto-Bäcklund transformations can be identified using HC-projections, only. The current section discusses this application.

So far, symmetry groups have only been used to reduce extended equations, via a search for group-invariant solutions, to obtain new equations related by HC-projections. Use will now be made of the most important property of symmetry groups — that they map solutions of a differential equation into solutions of the same differential equation. In particular, symmetries of HC-projected problems will be used to obtain new solution-generating techniques for the equations with which the projected problems are associated. Given a solution to a differential equation, one constructs the corresponding solution to some HC-projected problem. One then applies a symmetry group of that projected problem and foliates the new solution, obtaining a multi-parameter family of solutions to the original differential equation. The aim is to do this in such a way that the old and new solutions to the differential equation are not related by any symmetry group of that equation, thus introducing a genuinely new technique for generating solutions.

It is helpful to know exactly what symmetries of an HC-projected problem are inherited from the parent equation. Recall from Proposition 3.4 that every symmetry group of a system of differential equations yields a group of symmetries of each r-extended problem. A slightly less general result holds for HC-projected problems, which requires the introduction of some terminology before it can be formally stated. Let G be a Lie group with subgroup H. The subgroup of G defined by

$$N[H] = \{g \in G : ghg^{-1} \in H, \ \forall h \in H\}$$

is called the *normalizer of H in G* and is the largest subgroup of G admitting H as a normal subgroup.

Proposition 3.11 *Let G denote the symmetry group of a system $\Delta[u] = 0$ of n-th order differential equations on $M \subseteq X \times U$ and let H be a subgroup of G. Suppose that $\Phi : N \to M^{(n)}$ is a solution of $\Pi_H(\Delta)$. Then, for all $g \in N[H]$ for which $\mathrm{pr}^{(n)}g \cdot \Phi(N)$ is defined, $\mathrm{pr}^{(n)}g \cdot \Phi : N \to M^{(n)}$ is also a solution of $\Pi_H(\Delta)$.*

PROOF: By Proposition 3.4, $\mathrm{pr}^{(n)}g \cdot \Phi : N \to M^{(n)}$ is certainly a solution of the $(\dim H)$-extended problem associated with Δ. Let $y \in N$ be arbitrary. Then, since $\Phi(N)$ is locally $\mathrm{pr}^{(n)}H$-invariant, there exists a neighbourhood of the identity in H such that $\mathrm{pr}^{(n)}h \cdot \Phi(y) \in \Phi(N)$ for all h in that neighbourhood. Let $h' = ghg^{-1}$, which is an element of H since $g \in N[H]$. Then, using the fact that

$$(\mathrm{pr}^{(n)}g_1)(\mathrm{pr}^{(n)}g_2) = \mathrm{pr}^{(n)}(g_1g_2),$$

it follows that

$$\mathrm{pr}^{(n)}h' \cdot \mathrm{pr}^{(n)}g \cdot \Phi(y) = \mathrm{pr}^{(n)}g \cdot \mathrm{pr}^{(n)}h \cdot \Phi(y) \in \mathrm{pr}^{(n)}g \cdot \Phi(N)$$

for all h' in some neighbourhood of the identity in H. Thus $\mathrm{pr}^{(n)}g \cdot \Phi(N)$ is also locally $\mathrm{pr}^{(n)}H$-invariant and the proof is complete. $\square$

Continuing with the notation of Proposition 3.11, the symmetry group of the HC-projected problem $\Pi_H(\Delta)$ contains $N[H]$ as a subgroup. When an explicit parametrization is chosen, it follows that the symmetry group of the H-induced HC-projected equation associated with Δ admits a subgroup isomorphic to the quotient group $N[H]/H$.

Unfortunately, but not surprisingly, the symmetry group $N[H]$ of $\Pi_H(\Delta)$ contributes nothing new. Suppose that $\Phi : N \rightarrow M^{(n)}$ is a solution to Δ, so that $\mathrm{pr}^{(n)}H \cdot \Phi(N)$ is the corresponding solution to $\Pi_H(\Delta)$, assuming that such a solution exists. If $g \in N[H]$, then the new solution of $\Pi_H(\Delta)$ must be

$$\mathrm{pr}^{(n)}g \cdot (\mathrm{pr}^{(n)}H \cdot \Phi(N)) = \mathrm{pr}^{(n)}H \cdot \mathrm{pr}^{(n)}g \cdot \Phi(N),$$

since H is a normal subgroup of $N[H]$. Now, $\mathrm{pr}^{(n)}h \cdot \mathrm{pr}^{(n)}g \cdot \Phi(N)$ is clearly a solution to Δ for all $h \in H$, so that the second part of Proposition 3.4 implies that the leaves of the foliation of $\mathrm{pr}^{(n)}H \cdot \mathrm{pr}^{(n)}g \cdot \Phi(N)$ are just $\mathrm{pr}^{(n)}h \cdot \mathrm{pr}^{(n)}g \cdot \Phi(N)$ for all values of $h \in H$. That is, the new solutions to Δ are simply $\mathrm{pr}^{(n)}(hg) \cdot \Phi(N)$, and could have been obtained by elementary means, without the need to involve the HC-projected problem at all.

Consequently, if one wishes to use $\Pi_H(\Delta)$ to extend the capacity to generate solutions of Δ, it is essential that the symmetry group of $\Pi_H(\Delta)$, denoted by K hereafter, does not equal $\mathrm{pr}^{(n)}N[H]$. When this condition is satisfied, $\Pi_H(\Delta)$ will be said to possess *additional symmetries.* Schematically, the process to be invoked is

$$\begin{array}{ccc} \text{Initial solution to } \Delta & & \text{New solution to } \Delta \\ \downarrow & & \downarrow \\ \text{Solution to } \Pi_H(\Delta) & \stackrel{k \in K \setminus \mathrm{pr}^{(n)}N[H]}{\longrightarrow} & \text{Solution to } \Pi_H(\Delta) \end{array}$$

where the left hand arrow represents projecting a solution of Δ onto a solution of $\Pi_H(\Delta)$. The central arrow indicates application of an element k of $K \setminus \mathrm{pr}^{(n)}N[H]$

and the right hand arrow represents the process of lifting this new solution from $\Pi_H(\Delta)$ to Δ. The resulting family of solutions to Δ involves up to $\dim K + \dim H$ parameters and will not, in general, be related to the original solution of Δ by any symmetry group of that equation.

One would like to repeat this procedure, this time starting with the new solution, and perhaps obtain another solution of Δ. Suppose that an element g of the symmetry group of Δ is first applied to the new solution before the next application of the method above. (If one wants to proceed directly, then take $g = e$.) Let $\{\Phi_\gamma : N \longrightarrow M^{(n)} : \gamma \in \Gamma\}$ denote the collection of solutions already obtained, so that

$$\bigcup_{\gamma\in\Gamma} \Phi_\gamma(N) = k \cdot (\mathrm{pr}^{(n)}H \cdot \Phi(N)).$$

Then

$$\bigcup_{\gamma\in\Gamma} \mathrm{pr}^{(n)}g \cdot \Phi_\gamma(N) = \mathrm{pr}^{(n)}g \cdot k \cdot (\mathrm{pr}^{(n)}H \cdot \Phi(N))$$

are the new solutions to Δ and the corresponding solutions to the HC-projected problem are

$$\mathrm{pr}^{(n)}H \cdot \mathrm{pr}^{(n)}g \cdot k \cdot (\mathrm{pr}^{(n)}H \cdot \Phi(N)).$$

Subsequent application of an element k' of the symmetry group K of $\Pi_H(\Delta)$ yields the new solution

$$k' \cdot (\mathrm{pr}^{(n)}H \cdot \mathrm{pr}^{(n)}g \cdot k \cdot (\mathrm{pr}^{(n)}H \cdot \Phi(N)))$$

to that problem. Foliating this submanifold of $M^{(n)}$ yields another multi-parameter family of solutions to Δ. To clarify matters, consider the following summary:

$$\begin{array}{ccccccc}
\Delta & & \Delta & \xrightarrow{\mathrm{pr}^{(n)}g} & \Delta & & \Delta \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\Pi_H(\Delta) & \xrightarrow{k} & \Pi_H(\Delta) & & \Pi_H(\Delta) & \xrightarrow{k'} & \Pi_H(\Delta)
\end{array}$$

There is one absolutely crucial point to notice here. If $g \in N[H]$ then the new solutions to $\Pi_H(\Delta)$ are

$$k' \cdot \mathrm{pr}^{(n)}g \cdot (\mathrm{pr}^{(n)}H \cdot k \cdot (\mathrm{pr}^{(n)}H \cdot \Phi(N))) = k' \cdot \mathrm{pr}^{(n)}g \cdot k \cdot (\mathrm{pr}^{(n)}H \cdot \Phi(N)),$$

due to the $\mathrm{pr}^{(n)}H$-invariance of $k \cdot (\mathrm{pr}^{(n)}H \cdot \Phi(N))$. From Proposition 3.11, $\mathrm{pr}^{(n)}g$ is a symmetry of the HC-projected problem, so that $k' \cdot \mathrm{pr}^{(n)}g \cdot k$ is also a symmetry of $\Pi_H(\Delta)$. In this case, the above synopsis reduces to

$$\begin{array}{ccc} \Delta & & \Delta \\ \downarrow & & \downarrow \\ \Pi_H(\Delta) & \xrightarrow{k'\cdot \mathrm{pr}^{(n)} g\cdot k} & \Pi_H(\Delta) \end{array}$$

indicating that the whole process could have been performed using just one application of the initial method. Thus, if the second iteration is to be worthwhile, it is essential that g be chosen from $G \setminus N[H]$, yielding the second component of the auto-Bäcklund transformations introduced here. This is shown schematically below:

$$\begin{array}{ccc} \text{Solution to } \Delta & \xrightarrow{g\in G\setminus N[H]} & \text{Solution to } \Delta \\ \downarrow & & \downarrow \\ \text{Initial solution to } \Pi_H(\Delta) & & \text{New solution to } \Pi_H(\Delta) \end{array}$$

By alternating the two techniques starting from a seed solution it is possible to construct an indefinite sequence of solutions to Δ, solutions which are not normally obtainable by classical symmetry group methods.

For the second component to be possible, it is necessary that $G \neq N[H]$. Equivalently, H is not a normal subgroup of G. When this is the case and the H-induced HC-projected problem possesses additional symmetries, Δ and $\Pi_H(\Delta)$ are said to have *inequivalent symmetry groups*. An example of such a situation follows. It constructs an auto-Bäcklund transformation for the Harry Dym equation.

Example 3.12 The Harry Dym equation

$$0 = v_t + v^3 v_{yyy}$$

first appeared in work by Kruskal [56]. For a review of this equation, including several derivations, consult reference [40]. The symmetry group of the Harry Dym equation is five-dimensional and includes among its infinitesimal generators the vector fields

$$\mathbf{v}_1 = \partial_y, \quad \mathbf{v}_2 = y^2\partial_y + 2yv\partial_v.$$

Parametrizing solutions of the associated one-extended problem by $v_y = w(y,t,v)$ and restricting to solutions invariant under the group generated by the prolongation of $\mathbf{v}_1$ leads to the, admittedly unnatural, *Ansatz* $w(y,t,v) = 2u(x,t)$ where $x =$

$\frac{1}{2}\log v$. The $\exp(a\mathbf{v}_1)$-induced HC-projected equation associated with the Harry Dym equation is the third order differential equation

$$0 = u_t + u^3 u_{xxx} + 3u^2 u_x u_{xx} - 4u^3 u_x. \tag{3.18}$$

A solution of this equation can be lifted to a one-parameter family of solutions of the Harry Dym equation by solving the system

$$v_y = 2u, \quad v_t = -2e^{2x} u(uu_{xx} + u_x^2 - 2uu_x),$$

after making the substitution $x = \frac{1}{2}\log v$.

The significance of this HC-projection is that equation (3.18) has a discrete symmetry $k : (x, t, u) \mapsto (-x, t, -u)$ which is not the projection of any symmetry of the Harry Dym equation. To see why this is so, suppose that $\tilde{k} : (y, t, v) \mapsto (\tilde{y}, \tilde{t}, \tilde{v})$ is such a symmetry. Then clearly $\tilde{t} = t$, $\tilde{v} = 1/v$ and the prolongation of $\tilde{k}$ must be such that $\tilde{v}_{\tilde{y}} = -v_y$, otherwise $\tilde{k}$ cannot possibly project onto k. However, if $\tilde{y} = f(y, t, v)$ for some smooth function f defined on an open subset of (y, t, v)-space then, by prolonging the group action in the usual way, one finds that $\tilde{v}_{\tilde{y}}$ is determined by the equation

$$0 = \tilde{v}_{\tilde{y}} \left(\frac{\partial f}{\partial y} + v_y \frac{\partial f}{\partial v} \right) + v^{-2} v_y,$$

whence

$$0 = v_y^2 \frac{\partial f}{\partial v} + v_y \left(\frac{\partial f}{\partial y} - v^{-2} \right),$$

an equation which clearly has no solution. Furthermore, the symmetry generator $\mathbf{v}_2$ of the Harry Dym equation generates a one-dimensional symmetry group $\exp(a\mathbf{v}_2)$ acting via

$$\exp(a\mathbf{v}_2) : (y, t, v) \mapsto \left(\frac{y}{1 - ay}, t, \frac{v}{(1 - ay)^2} \right)$$

whenever $a < 1/y$. Equivalently, if $v(y, t)$ is a solution of the Harry Dym equation, then so is

$$\tilde{v}_a(y, t) = (1 + ay)^2 \cdot v\left(\frac{y}{1 + ay}, t \right). \tag{3.19}$$

This group action does not project onto a symmetry of equation (3.18).

As described above, one can combine the symmetries k and $\exp(a\mathbf{v}_2)$ to obtain an auto-Bäcklund transformation for the Harry Dym equation. The way in which these symmetries are to be implemented is shown below:

$$\begin{array}{ccccccc} v_0(y,t) & \stackrel{\exp(a\mathbf{v}_2)}{\longrightarrow} & v_1(y,t) & & v_2(y,t) & \stackrel{\exp(-a\mathbf{v}_2)}{\longrightarrow} & v_3(y,t) \\ & & \downarrow & & \downarrow & & \\ & & u_1(x,t) & \stackrel{k}{\longrightarrow} & u_2(x,t) & & \end{array}$$

One can say that $v_0(y,t)$ and $v_3(y,t)$ are solutions of the Harry Dym equation related by this auto-Bäcklund transformation. This decomposition does not fit, exactly, that described before this example, the difference being the initial application of $\exp(a\mathbf{v}_2)$, the inverse of the symmetry of the Harry Dym equation which will conclude the transformation. While it is not necessary to perform this first step, it is included here as many of the well known auto-Bäcklund transformations factorize in this way. See Example 5.22 for a similar decomposition of one of the most famous auto-Bäcklund transformations of all — that for the KdV equation.

As an example of the application of this auto-Bäcklund transformation for the Harry Dym equation, take $v_0(y,t) = 1$ as the seed solution. Then, from equation (3.19), $v_1(y,t) = (1+ay)^2$, so that the corresponding solution to equation (3.18) is given parametrically by

$$x = \log(1+ay), \quad u = a(1+ay).$$

That is, $u_1(x,t) = a\exp x$ and the new solution to the HC-projected equation is thus $u_2(x,t) = -a\exp(-x)$. The system of first order equations which describes the corresponding family of solutions to the Harry Dym equation is easily shown to be

$$v_y = \frac{-2a}{\sqrt{v}}, \quad v_t = \frac{8a^3}{\sqrt{v}},$$

and has general solution

$$v_2(y,t) = (12a^3t - 3ay + b)^{2/3}.$$

The step applying $\exp(-a\mathbf{v}_2)$ yields

$$v_3(y,t) = (1-ay)^{4/3}\left((12a^3t+b)(1-ay) - 3ay\right)^{2/3},$$

which is thus the solution obtained from the trivial seed solution $v_0 \equiv 1$ using this auto-Bäcklund transformation. One could now repeat this procedure, letting v_0 take the value of v_3. Provided that the parameter involved in the initial and final steps is different from that used in the first iteration, another, more complicated solution will result. This solution, which is not presented here, cannot be written explicitly but must instead by described parametrically.

The auto-Bäcklund transformation described in this example is closely related to the "reciprocal Bäcklund transformation" of Rogers and Wong [77]. Their transformation coincides with the mapping $v_1(y,t) \mapsto v_2(y,t)$ described above. Suppose that $v(y,t)$ is a solution to the Harry Dym equation and that $\tilde{y} = \tilde{y}(y,t)$, $\tilde{t} = \tilde{t}(y,t)$ and $\tilde{v} = \tilde{v}(y,t)$ describe the new solutions to that equation resulting from application of the discrete symmetry k of equation (3.18). As mentioned earlier, $\tilde{t} = t$, $\tilde{v} = 1/v$ and $\tilde{v}_{\tilde{y}} = -v_y$ are known immediately. Therefore, from equations (2.2) it follows that

$$-v^{-2}v_y = \frac{\partial \tilde{v}}{\partial y} = -v_y\frac{\partial \tilde{y}}{\partial y},$$

whence

$$\frac{\partial \tilde{y}}{\partial y} = v^{-2}. \tag{3.20}$$

As a consequence of this result, other derivatives of $\tilde{v}(\tilde{y},\tilde{t})$ can be calculated. Appropriate ones are

$$\tilde{v}_{\tilde{y}\tilde{y}} = -v^2 v_{yy}, \quad \tilde{v}_{\tilde{y}\tilde{y}\tilde{y}} = -v^4 v_{yyy} - 2v^3 v_y v_{yy},$$

and

$$\tilde{v}_{\tilde{t}} = -\tilde{v}^3 \tilde{v}_{\tilde{y}\tilde{y}\tilde{y}} = -v^{-2}v_t + 2v_y v_{yy},$$

so that

$$-v^{-2}v_t = \frac{\partial \tilde{v}}{\partial t} = (-v^{-2}v_t + 2v_y v_{yy})\frac{\partial \tilde{t}}{\partial t} - v_y\frac{\partial \tilde{y}}{\partial t}.$$

Since $\tilde{t} = t$ it follows that

$$\frac{\partial \tilde{y}}{\partial t} = 2v_{yy} \tag{3.21}$$

and the new solution to the Harry Dym equation is described parametrically by $\tilde{y} = \tilde{y}(y,t)$, $\tilde{t} = t$ and $\tilde{v} = 1/v(y,t)$ where $\tilde{y}$ is determined by equations (3.20) and (3.21). This is precisely the transformation obtained by Rogers and Wong. It is called a reciprocal transformation because applying it twice yields the identity transformation, reflecting the fact that the discrete symmetry group generated by k has order two. The continuous symmetry group $\exp(a\mathbf{v}_2)$ of the Harry Dym equation which allows one to apply the auto-Bäcklund transformation repeatedly was overlooked by Rogers and Wong. This auto-Bäcklund transformation is studied further in Section 4.5 while a genuinely new auto-Bäcklund transformation for the Harry Dym equation is presented in Example 5.24.

Finally, the construction of the transformation $(y,t,v) \mapsto (\tilde{y}(y,t), \tilde{t}(y,t), \tilde{v}(y,t))$ does not contradict the claim made earlier in this example that the symmetry k of equation (3.18) is not the projection of a symmetry of the Harry Dym equation. The requirement to solve equations (3.20) and (3.21) indicates that the transformation $(y,t,v) \mapsto (\tilde{y}(y,t), \tilde{t}(y,t), \tilde{v}(y,t))$ is not a diffeomorphism of the space with coordinates (y,t,v), as was required earlier, but rather an association between certain two-dimensional submanifolds of this space. □

One sees from this example that knowledge of HC-projections can significantly extend the solution-generating techniques available for some differential equations. Given a differential equation Δ with symmetry group G one requires a subgroup H of G such that Δ and $\Pi_H(\Delta)$ have inequivalent symmetry groups. That is, the symmetry group of $\Pi_H(\Delta)$ must not equal $N[H]$ and, in addition, $N[H] \neq G$. The second condition is easily guaranteed by restricting attention to subgroups of G which are not normal. Unfortunately, the first condition is much harder to satisfy, as there is currently no way to predict which subgroups of G lead to HC-projected problems with additional symmetry. The appearance of additional symmetries is not unique to HC-projected problems and can also occur while searching for ordinary group invariant solutions to a differential equation. For examples of extra symmetries appearing after a group reduction see Example 3.5(b) of [72].

In the absence of a method for predicting when extra symmetries arise, one should perform the following steps when confronted with a differential equation Δ.

1. Calculate the symmetry algebra of Δ.

2. Construct an optimal system of subalgebras for this symmetry algebra.

3. For each subalgebra of this optimal system, determine the symmetry algebra of the corresponding HC-projected equation.

There are two important points to notice about this procedure.

The first point involves a serious shortcoming. Throughout the above procedure use has been made of the symmetry algebra, and its subalgebras, of the original differential equation and its various HC-projections. This is usually acceptable, as one is mainly interested in symmetries of a differential equation which can be continuously connected to the trivial symmetry, so that infinitesimal techniques are appropriate. However, often when dealing with auto-Bäcklund transformations one of the component symmetry groups is continuous, such as $\exp(a\mathbf{v}_2)$ in Example 3.12, while the other component is discrete — k, in the case of the above example. Since discrete symmetries can be extremely difficult to find, it is usually not possible to construct the full symmetry group of Δ and its HC-projected equations. Instead, one must hope that any discrete symmetry becomes almost trivial. It was fortunate that in Example 3.12 coordinates were chosen for the HC-projected equation such that the discrete symmetry took the easily identifiable form $k : (x, t, u) \mapsto (-x, t, -u)$. Even a simple change of coordinates, such as the natural choice of t and $v = \exp(2x)$ as independent variables, may have obscured this symmetry. Section 5.5 will develop a technique for constructing Bäcklund transformations based purely on infinitesimal techniques.

Secondly, if one is prepared to accept a transformation which stops after a single iteration then it is possible to allow H to be a normal subgroup of G. The additional symmetry of $\Pi_H(\Delta)$ can still be used to generate a new solution of Δ. However, the symmetry of Δ in $G \backslash N[H]$ which allows the transformation to be applied repeatedly does not exist, hence the process terminates.

Chapter 4

More general transformations

The HC-projections introduced in Section 3.3 are shown to be closely related to special types of Wahlquist-Estabrook prolongation in Section 4.1. It is demonstrated that a differential equation is equivalent to some Wahlquist-Estabrook prolongation of each HC-projection of that equation. A symmetry based characterization of exactly which Wahlquist-Estabrook prolongations of a given differential equation arise in this way is also presented. In Section 4.2 a flat connection on a principal fibre bundle is constructed for every solution to an HC-projected problem associated with a given differential equation. This result is compared with the connections on fibre bundles usually derived from Wahlquist-Estabrook prolongations. The next two sections examine the ways in which complicated HC-projections can be broken down into simpler transformations. Section 4.3 determines when an HC-projection can be decomposed into a sequence of lower order HC-projections and, in Section 4.4, this is generalized to include component transformations which are not necessarily HC-projections. These new transformations are analogues of the well known Miura transformation, which relates the KdV and mKdV equations in a manner similar, but not identical, to HC-projections. Properties of these "M-projections" are studied, but techniques for their construction are delayed until Chapter 5. Finally, the extended example which constitutes Section 4.5 demonstrates that HC- and M-projections are associated with some widely studied integrable equations. Possibilities for a group-theoretic interpretation of many of the properties associated with these equations are suggested.

4.1 Wahlquist-Estabrook prolongations and HC-projections

HC-projections and the prolongation process of Wahlquist and Estabrook work in essentially opposite directions. For instance, Example 2.4 recovered the heat equation as a prolongation of Burgers' equation involving a single pseudopotential, while, as demonstrated in Example 3.8, Burgers' equation arises as an HC-projected equation associated with the heat equation.

This section begins by proving that a differential equation is equivalent to some Wahlquist-Estabrook prolongation of any HC-projection of that equation. Let Δ be a differential equation and suppose that G is an r-dimensional symmetry group of Δ. The construction of an associated G-induced HC-projected equation begins with an r-extended equation associated with Δ. Suppose that this extended equation involves independent variables $x = (x^1, \ldots, x^{p+r})$ and dependent variables $u = (u^1, \ldots, u^q)$. G extends to a symmetry group $\tilde{G}$ of this equation which acts on the space with coordinates (x, u) and is assumed to have r-dimensional orbits. Let the vector fields $\{\mathbf{v}_a : a = 1, \ldots, r\}$ be a basis of infinitesimal generators of the $\tilde{G}$-action. The construction of $\tilde{G}$-invariant solutions to the extended equation, following the method described in Section 2.3, leads to a G-induced HC-projected equation associated with Δ which will be denoted by $\Pi_G(\Delta)$. This equation involves independent variables $y = (y^1, \ldots, y^p)$ and dependent variables $v = (v^1, \ldots, v^q)$. Also introduced in the reduction process are the parametric variables $\hat{x} = (\hat{x}^1, \ldots, \hat{x}^r)$, yielding alternative local coordinates $(y, v, \hat{x})$ for (x, u)-space.

Solutions of $\Pi_G(\Delta)$ are lifted to solutions of Δ by foliating the corresponding solutions to the r-extended problem associated with Δ. Let $\{\theta^a : a = 1, \ldots, r\}$ be a basis for the module of one-forms which define this foliation. Each θ^a can be described in terms of the variables occurring in the extended equation and can thus be expressed in terms of $(y, v^{(n-1)}, \hat{x})$ for some positive integer n. It will be assumed that the matrix $[\theta^a(\mathbf{v}_b)]$ is invertible everywhere on this solution. Such an assumption amounts to a transversality requirement of the group action and will be pursued further in Section 4.2. From the construction of the coordinates $(y, v, \hat{x})$ it is obvious that

$$\mathbf{v}_a = \sum_{b=1}^{r} d\hat{x}^b(\mathbf{v}_a) \cdot \partial_{\hat{x}^b}, \quad a = 1, \ldots, r,$$

whence

$$\theta^a(\mathbf{v}_b) = \theta^a\left(\sum_{c=1}^{r} d\hat{x}^c(\mathbf{v}_b) \cdot \partial_{\hat{x}^c}\right) = \sum_{c=1}^{r}(d\hat{x}^c(\mathbf{v}_b))(\theta^a(\partial_{\hat{x}^c})), \quad a, b = 1, \ldots, r. \tag{4.1}$$

Regularity of the $\tilde{G}$-action implies that the matrix $[d\hat{x}^a(\mathbf{v}_b)]$ is invertible everywhere, so that equations (4.1) imply that the matrix $[\theta^a(\partial_{\hat{x}^c})]$ must also be invertible. Consequently, the system of equations

$$\sum_{c=1}^{r} \theta^a(\partial_{\hat{x}^c}) f_b^c = \delta_b^a, \quad a, b = 1, \ldots, r,$$

has a unique solution and the one forms

$$\omega^a = \sum_{b=1}^{r} f_b^a \theta^b, \quad a = 1, \ldots, r,$$

provide an alternative basis for the module of one-forms involved in the lifting process. Here δ_b^a denotes the Kronecker delta symbol.

The one-forms $\{\omega^a : a = 1, \ldots, r\}$ describe a Wahlquist-Estabrook prolongation of $\Pi_G(\Delta)$ which is equivalent to Δ. By their construction they satisfy $\omega^a(\partial_{\hat{x}^b}) = \delta_b^a$ and must therefore have the form

$$\omega^a = d\hat{x}^a - \sum_{i=1}^{p} F_i^a(y, v^{(n-1)}, \hat{x}) dy^i, \quad a = 1, \ldots, r,$$

for suitable smooth functions F_i^a. On solutions to $\Pi_G(\Delta)$ the ideal generated by $\{\omega^a : a = 1, \ldots, r\}$ is closed under exterior differentiation, so that

$$0 = \sum_{1 \le i < j \le p} \left(D_{y^i}(F_j^a) - D_{y^j}(F_i^a) + \sum_{b=1}^{r} \left(F_i^b \frac{\partial F_j^a}{\partial \hat{x}^b} - F_j^b \frac{\partial F_i^a}{\partial \hat{x}^b} \right) \right) dy^i \wedge dy^j,$$

for all $a = 1, \ldots, r$. That is, the prolongation equations

$$\hat{x}_i^a = F_i^a(y, v^{(n-1)}, \hat{x}), \quad i = 1, \ldots, p, \;\; a = 1, \ldots, r, \tag{4.2}$$

define a Wahlquist-Estabrook prolongation of $\Pi_G(\Delta)$. Obviously the prolonged system of equations comprising $\Pi_G(\Delta)$ and equations (4.2) is equivalent to the original differential equation Δ — one need only consider the foliation of solutions to the

G-induced HC-projected problem determined by equations (4.2) to see that this is so.

Furthermore, this prolongation of $\Pi_G(\Delta)$ does not feature any redundant pseudopotentials. Suppose that the function $\theta(y, v^{(n-1)}, \hat{x})$ describes such redundancy. Then, on solutions to $\Pi_G(\Delta)$, θ is constant and

$$0 = d\theta = \sum_{a=1}^{r} \frac{\partial\theta}{\partial\hat{x}^a} d\hat{x}^a + \sum_{i=1}^{p} (D_{y^i}\theta) dy^i.$$

Thus

$$\sum_{a=1}^{r} \frac{\partial\theta}{\partial\hat{x}^a}\omega^a = \sum_{a=1}^{r} \frac{\partial\theta}{\partial\hat{x}^a}\left(d\hat{x}^a - \sum_{i=1}^{p} F_i^a dy^i\right) = -\sum_{i=1}^{p} (\tilde{D}_{y^i}\theta) dy^i = 0,$$

and the restriction of the contact module, which is spanned by $\{\omega^a : a = 1, \ldots, r\}$, has dimension less than r. This contradiction proves that no such redundancy can exist.

Analysis of the first order HC-projections of the heat equation is continued in the following example, which interprets the results of Example 3.8 in light of the preceding discussion.

Example 4.1 For each equation derived as a first order HC-projection of the heat equation in Example 3.8, a Wahlquist-Estabrook prolongation will be constructed following the method discussed above. Once again, the construction will be given in detail only for the first of these equations.

(a) In terms of the principal coordinates y, z and u and the parametric variable t used in the reduction of the first extension of the heat equation which is induced by the vector field $a\mathbf{v}_3 + \mathbf{v}_4$, the one-form generating the restriction of the contact module is

$$\eta = -\tfrac{1}{2}t^{(a-2)/2}(2u_y + 2uu_z + yu - az)dt + t^{a/2}(dz - u dy).$$

Therefore, the required prolongation form is

$$\begin{aligned}\omega &= \frac{-2t^{(2-a)/2}}{2(u_y + uu_z) + yu - az} \cdot \eta \\ &= dt + \frac{2tu}{2(u_y + uu_z) + yu - az} dy - \frac{2t}{2(u_y + uu_z) + yu - az} dz,\end{aligned}$$

which defines a Wahlquist-Estabrook prolongation of equation (3.8) featuring a pseudopotential t defined by the system of equations

$$t_y = \frac{-2tu}{2(u_y + uu_z) + yu - az}, \quad t_z = \frac{2t}{2(u_y + uu_z) + yu - az}. \tag{4.3}$$

If $u(y,z)$ is a solution of equation (3.8) and $t(y,z)$ satisfies equations (4.3) then the submanifold of $M = X \times U = \mathbb{R}^2 \times \mathbb{R}^1$ given by

$$x = (t(y,z))^{1/2} y, \quad t = t(y,z), \quad v = (t(y,z))^{a/2} z, \tag{4.4}$$

describes a solution of the heat equation. These equations are obtained by writing the variables involved in the heat equation in terms of the parametric and principal variables used in the reduction process. Further, from equations (4.3) one finds that $u = -t_y t_z^{-1}$ and that $t(y,z)$ must satisfy the second order differential equation

$$0 = 2(t_{yy} t_z^2 - 2t_y t_z t_{yz} + t_y^2 t_{zz}) + y t_y t_z^2 + 2t t_z^2 + az t_z^3.$$

This equation is a modified version of equation (3.8) and is equivalent to the heat equation under the invertible change of coordinates described by equations (4.4).

(b) Taking t as parametric variable in the projection induced by the symmetry group with infinitesimal generator $\mathbf{v}_2 + b\mathbf{v}_3 + \mathbf{v}_6$ leads to a Wahlquist-Estabrook prolongation of the HC-projected equation of case (b) described by the system of equations

$$t_y = \frac{-u(1+4t^2)}{u_y + u u_z + y^2 z - bz}, \quad t_z = \frac{1+4t^2}{u_y + u u_z + y^2 z - bz}.$$

Once more $u = -t_y t_z^{-1}$, but this time the modified equation is

$$0 = t_{yy} t_z^2 - 2t_y t_z t_{yz} + t_y^2 t_{zz} + t_z^2(1+4t^2) - t_z^3(y^2 z - bz).$$

Again, it can be rewritten as the heat equation following an appropriate coordinate change.

(c) The infinitesimal symmetry generator $\mathbf{v}_2 - \mathbf{v}_5$ leads to a reduction of the first extension of the heat equation with t as parametric variable and Wahlquist-Estabrook prolongation defined by

$$t_y = \frac{-u}{u_y + u u_z - yz}, \quad t_z = \frac{1}{u_y + u u_z - yz}.$$

Consequently, $u = -t_y t_z^{-1}$ and the modified equation is

$$0 = t_{yy} t_z^2 - 2t_y t_z t_{yz} + t_y^2 t_{zz} + t_z^2 + yz t_z^3.$$

It is equivalent to the heat equation under an invertible change of coordinates.

(d) Taking t as parametric variable, the infinitesimal symmetry generator $\mathbf{v}_2+c\mathbf{v}_3$ of the heat equation yields an HC-projected equation with Wahlquist-Estabrook prolongation given by the equations

$$t_x = \frac{-u}{u_x + uu_y - cy}, \quad t_y = \frac{1}{u_x + uu_y - cy}.$$

Since $u = -t_x t_y^{-1}$, it follows that the modified equation is

$$0 = t_{xx}t_y^2 - 2t_x t_y t_{xy} + t_x^2 t_{yy} + t_y^2 + cyt_y^3.$$

Again, it is just the heat equation in disguise.

(e) The projection of the heat equation induced by the symmetry group with generator $\mathbf{v}_1$ has prolongation equations

$$x_v = 1/u, \quad x_t = -u_v,$$

involving the parametric variable x. After observing that $u = x_v^{-1}$ it follows that the modified equation is

$$x_t = x_v^{-2} x_{vv}.$$

The relationships between this equation and the heat equation, to which it transforms under an invertible change of coordinates, and also to the relevant projection of the heat equation have been known for many years [78].

(f) Burgers' equation is obtained by constructing an HC-projected equation induced by the symmetry group generated by $\mathbf{v}_3$. The parametric variable v involved in the projection is a pseudopotential of Burgers' equation, defined by

$$v_x = uv, \quad v_t = (u_x + u^2)v.$$

Eliminating u using $u = v^{-1}v_x$ recovers the heat equation as a modification of Burgers' equation.

(g) Finally, consider the projected equation associated with the symmetry group with generator $\mathbf{v}_\theta$. Once again the parametric variable is v, which can be interpreted as a pseudopotential defined by the equations

$$v_x = \theta^{-1}\theta_x v + u, \quad v_t = \theta^{-1}\theta_{xx} v + \theta^{-1}\theta_x u + u_x.$$

It follows that $u = v_x - \theta^{-1}\theta_x v$, so that the relevant modified equation is just the heat equation.

	Projected equation	Prolongation equations
(a)	$0 = 2(u_{yy} + 2uu_{yz} + u^2u_{zz}) + yu_y + u + a(zu_z - u)$	$t_y = \dfrac{-2tu}{2(u_y + uu_z) + yu - az}$, $t_z = \dfrac{2t}{2(u_y + uu_z) + yu - az}$
(b)	$0 = u_{yy} + 2uu_{yz} + u^2u_{zz} + (zu_z - u)(b - y^2) + 2yz$	$t_y = \dfrac{-u(1+4t^2)}{u_y + uu_z + y^2z - bz}$, $t_z = \dfrac{1+4t^2}{u_y + uu_z + y^2z - bz}$
(c)	$0 = u_{yy} + 2uu_{yz} + u^2u_{zz} + yzu_z - yu - z$	$t_y = \dfrac{-u}{u_y + uu_z - yz}$, $t_z = \dfrac{1}{u_y + uu_z - yz}$
(d)	$0 = u_{xx} + 2uu_{xy} + u^2u_{yy} + c(yu_y - u)$	$t_x = \dfrac{-u}{u_x + uu_y - cy}$, $t_y = \dfrac{1}{u_x + uu_y - cy}$
(e)	$u_t = u^2u_{vv}$	$x_v = 1/u, \quad x_t = -u_v$
(f)	$u_t = u_{xx} + 2uu_x$	$v_x = uv, \quad v_t = (u_x + u^2)v$
(g)	$u_t = u_{xx} + 2\theta^{-2}(\theta\theta_{xx} - \theta_x^2)u$	$v_x = \theta^{-1}\theta_x v + u$, $v_t = \theta^{-1}\theta_{xx}v + u_x + \theta^{-1}\theta_x u$

	Modified equation
(a)	$0 = 2(t_{yy}t_z^2 - 2t_yt_zt_{yz} + t_y^2t_{zz}) + yt_yt_z^2 + 2tt_z^2 + azt_z^3$
(b)	$0 = t_{yy}t_z^2 - 2t_yt_zt_{yz} + t_y^2t_{zz} + t_z^2(1+4t^2) - t_z^3(y^2z - bz)$
(c)	$0 = t_{yy}t_z^2 - 2t_yt_zt_{yz} + t_y^2t_{zz} + t_z^2 + yzt_z^3$
(d)	$0 = t_{xx}t_y^2 - 2t_xt_yt_{xy} + t_x^2t_{yy} + t_y^2 + cyt_y^3$
(e)	$x_t = x_v^{-2}x_{vv}$
(f)	$v_t = v_{xx}$
(g)	$v_t = v_{xx}$

Table 4.1: Wahlquist-Estabrook prolongations of equations recovering the heat equation

The results are summarized in Table 4.1 which lists the HC-projected equations, their prolongations and associated modified equations. It should be compared with Table 3.1 which presents much the same data, but in a form more appropriate to the HC-projection process. □

It has been shown that, subject to some technical requirements, any two equations related by an HC-projection are also related by a Wahlquist-Estabrook prolongation. However, the converse is not always true, as will be seen later when considering the prolongation of the KdV equation $0 = u_t + u_{xxx} + 12uu_x$ described by

$$v_x = v^2 + 2u, \quad v_t = -4uv^2 - 4u_x v - 2(u_{xx} + 4u^2).$$

This prolongation does not arise from an HC-projection of the combined system of equations onto the KdV equation.

There is a straightforward test to determine whether a Wahlquist-Estabrook prolongation can be interpreted as an HC-projection. It involves a special type of symmetry generator of the prolonged system of equations. These vector fields are defined below.

Definition 4.2 Let (Δ, Ξ) denote a Wahlquist-Estabrook prolongation to $M \times Y$ of a system of differential equations $\Delta[u] = 0$ on $M \subseteq X \times U$. Suppose that the vector field $\mathbf{v}$ on $M \times Y$ generates a symmetry group of (Δ, Ξ) and that $\pi_1 : M \times Y \to M$ denotes the trivial projection. If $(\pi_1)_* \mathbf{v} = 0$ then $\mathbf{v}$ is called an *internal symmetry generator* of (Δ, Ξ).

Let Y have coordinates $y = (y^1, \ldots, y^r)$ and suppose that the following property holds: If f is any smooth function on $M \times Y$ such that $\mathbf{v}(f) = 0$ for all internal symmetry generators $\mathbf{v}$, then

$$\frac{\partial f}{\partial y^a} = 0, \quad a = 1, \ldots, r.$$

Then (Δ, Ξ) is said to admit a *full internal symmetry group.* □

Vector fields satisfying Definition 4.2 are called internal symmetry generators because they are, in some sense, internal to the prolongation. The group actions which they generate affect only the pseudopotentials, leaving other variables unchanged. One easily proves that the set of internal symmetry generators is a subalgebra of the symmetry algebra of (Δ, Ξ), allowing one to speak of the *internal symmetry algebra* of a Wahlquist-Estabrook prolongation. The connected symmetry group generated by the internal symmetry algebra will be called the *internal symmetry group* of the prolongation.

Suppose that $Y = \mathbb{R}^r$ has coordinates $y = (y^1, \ldots, y^r)$ and that (Δ, Ξ) has full internal symmetry group. If $\{\mathbf{v}_a : a = 1, \ldots, s\}$ is a basis of internal symmetry generators with s finite and

$$\mathbf{v}_a = \sum_{b=1}^{r} h_a^b(x, u, y)\partial_{y^b}, \quad a = 1, \ldots, s,$$

then the matrix $[h_a^b]$ must have rank r. In particular, $s \geq r$. To see why this is so, consider the foliation of $M \times Y$ determined by the action of the internal symmetry group. From the full internal symmetry group property, leaves in this foliation are given by $\{(x, u)\} \times Y$ for each $(x, u) \in M$, and are thus r-dimensional. The following lemma shows that if, in addition to admitting a full internal symmetry group, (Δ, Ξ) is nondegenerate, then $r = s$.

Lemma 4.3 *Let (Δ, Ξ) denote a nondegenerate Wahlquist-Estabrook prolongation of an n-th order differential equation Δ to $M \times Y$, where $Y = \mathbb{R}^r$. If (Δ, Ξ) admits a full internal symmetry group then its internal symmetry algebra is r-dimensional.*

PROOF: Suppose that $M \subseteq X \times U$, where $X = \mathbb{R}^p$ and $U = \mathbb{R}^q$ have coordinates $x = (x^1, \ldots, x^p)$ and $u = (u^1, \ldots, u^q)$ respectively, and that the prolongation equations are

$$y_i^a = F_i^a(x, u^{(n-1)}, y), \quad i = 1, \ldots, p, \;\; a = 1, \ldots, r. \tag{4.5}$$

Let $\mathfrak{h}$ denote the internal symmetry algebra. The full internal symmetry group property implies that $\mathfrak{h}$ must be at least r-dimensional and there must exist internal symmetry generators

$$\mathbf{v}_a = \sum_{b=1}^{r} h_a^b(x, u, y)\partial_{y^b}, \quad a = 1, \ldots, r,$$

such that the matrix $[h_a^b]$ is invertible on $M \times Y$. Suppose that $\mathbf{u}$ is any other internal symmetry generator. Then there must exist smooth functions g^a on $M \times Y$ such that

$$\mathbf{u} = \sum_{a=1}^{r} g^a(x, u, y)\mathbf{v}_a.$$

Since the appropriate prolongation of $\mathbf{u}$ must leave equations (4.5) invariant, it follows that

$$\begin{aligned} 0 &= \mathrm{pr}^{(1)}\mathbf{u}(y_i^a - F_i^a) \\ &= \tilde{D}_{x^i}(g^b h_b^a) - \sum_{b,c=1}^{r} g^b h_b^c \frac{\partial F_i^a}{\partial y^c}, \quad i = 1, \ldots, p, \quad a = 1, \ldots, r, \end{aligned}$$

on $\mathcal{S}_\Delta \times Y$. Similarly,

$$0 = \tilde{D}_{x^i}(h_b^a) - \sum_{c=1}^{r} h_b^c \frac{\partial F_i^a}{\partial y^c}, \quad i = 1, \ldots, p, \quad a, b = 1, \ldots, r,$$

because each $\mathbf{v}_a$ is an internal symmetry generator, so that

$$0 = \sum_{b=1}^{r} \tilde{D}_{x^i}(g^b) \cdot h_b^a, \quad i = 1, \ldots, p, \quad a = 1, \ldots, r,$$

on $\mathcal{S}_\Delta \times Y$. As the matrix $[h_a^b]$ is invertible and the prolongation includes no redundant pseudopotentials, each g^b must be a constant on $\mathcal{S}_\Delta \times Y$, whence $\mathbf{u}$ is in the algebra spanned by $\{\mathbf{v}_a : a = 1, \ldots, r\}$. Thus $\mathfrak{h}$ is r-dimensional with basis $\{\mathbf{v}_a : a = 1, \ldots, r\}$. □

Existence or otherwise of a full internal symmetry group allows one to decide whether a Wahlquist-Estabrook prolongation arises from an HC-projection of some equation, as shown by the following theorem.

Theorem 4.4 *Let G denote a symmetry group of a differential equation Δ. Whenever the transversality assumption is satisfied, the Wahlquist-Estabrook prolongation $(\Pi_G(\Delta), \Xi)$ of $\Pi_G(\Delta)$ described at the beginning of this section is nondegenerate and admits a full internal symmetry group.*

Conversely, suppose that (Δ, Ξ) is a nondegenerate Wahlquist-Estabrook prolongation of a differential equation Δ with a full internal symmetry group. If H denotes the internal symmetry group then $\Pi_H(\Delta, \Xi) = \Delta$. That is, Δ arises as an H-induced HC-projected equation associated with (Δ, Ξ).

PROOF: Suppose that the system of equations $(\Pi_G(\Delta), \Xi)$ is described on $M \times Y$ with Y having coordinates $y = (y^1, \ldots, y^r)$. Following the discussion which began this section, choose a basis of prolongation forms $\{\omega^a : a = 1, \ldots, r\}$ such that

$$\omega^a = dy^a + \beta^a, \quad a = 1, \ldots, r,$$

where $\beta^a(\partial_{y^b}) = 0$ for all $a, b = 1, \ldots, r$. Let

$$\mathbf{v}_a = \sum_{b=1}^{r} \eta_a^b \partial_{y^b}, \quad a = 1, \ldots, r,$$

be an arbitrary basis of infinitesimal generators of the action of G on $M \times Y$, with appropriate smooth functions η_a^b on $M \times Y$. Transversality of the G-action means that the matrix with entries $\eta_a^b = \omega^b(\mathbf{v}_a)$ is invertible everywhere, with inverse $[\sigma_a^b]$, say. If the smooth function f on $M \times Y$ satisfies $\mathbf{v}_a(f) = 0$ for all $a = 1, \ldots, r$, then

$$\frac{\partial f}{\partial y^a} = \sum_{b=1}^{r} \sigma_a^b \mathbf{v}_b(f) = 0, \quad a = 1, \ldots, r.$$

Hence, $(\Pi_G(\Delta), \Xi)$ admits a full internal symmetry group. The absence of redundant pseudopotentials was shown earlier.

Conversely, suppose that Δ is an n-th order differential equation described on $M \subseteq X \times U$ where $X = \mathbb{R}^p$ and $U = \mathbb{R}^q$ have coordinates $x = (x^1, \ldots, x^p)$ and $u = (u^1, \ldots, u^q)$ respectively. Further suppose that Δ has been prolonged to $M \times Y$, with $Y = \mathbb{R}^r$ having coordinates $y = (y^1, \ldots, y^r)$, via the prolongation equations

$$y_i^a = F_i^a(x, u^{(n-1)}, y), \quad i = 1, \ldots, p, \quad a = 1, \ldots, r. \tag{4.6}$$

From Lemma 4.3, $\dim H = r$, and the construction of an H-induced HC-projected equation associated with (Δ, Ξ) is obtained by reducing a suitable r-extended equation. Take $\{x^1, \ldots, x^p, y^1, \ldots, y^r\}$ as independent variables for the r-extended equation associated with (Δ, Ξ). Solutions to the extended equation can be described by

$$\begin{aligned} u^\alpha &= v^\alpha(x, y), \quad \alpha = 1, \ldots, q, \\ y_i^a &= z_i^a(x, y), \quad i = 1, \ldots, p, \quad a = 1, \ldots, r, \end{aligned}$$

and similarly for the higher derivatives. If such solutions are to be invariant under the action of the internal symmetry group, it follows from the definition of a prolongation admitting a full internal symmetry group that

$$\frac{\partial v^\alpha}{\partial y^a} = 0, \quad \alpha = 1, \ldots, q, \quad a = 1, \ldots, r.$$

Consequently, $u^\alpha = v^\alpha(x)$ and the higher derivatives can be described by equations such as

$$u_i^\alpha = \frac{\partial v^\alpha}{\partial x^i} + \sum_{a=1}^{r} \frac{\partial v^\alpha}{\partial y^a} y_i^a = \frac{\partial v^\alpha}{\partial x^i}, \quad i = 1, \ldots, p, \quad \alpha = 1, \ldots, q,$$

and so on. Using the prolongation equations, one can write

$$z_i^a(x,y) = y_i^a = F_i^a(x, u^{(n-1)}, y) = F_i^a(x, v^{(n-1)}, y), \quad i = 1, \ldots, p, \quad a = 1, \ldots, r,$$

so that the H-induced HC-projected equation can be written solely in terms of the functions $\{v^\alpha(x) : \alpha = 1, \ldots, q\}$. One easily sees that the resulting equation is just $\Delta[v] = 0$, where $\Delta[u] = 0$ was the original differential equation. □

By restricting domains where appropriate, so that the transversality assumption holds everywhere, this theorem shows that a Wahlquist-Estabrook prolongation corresponds to some HC-projection if and only if the prolongation admits a full internal symmetry group and has no redundant pseudopotentials. A straightforward calculation confirms that the prolongation of the KdV equation described before Definition 4.2 admits no nonzero internal symmetry generators. By Theorem 4.4 it follows that the prolongation cannot be derived from an HC-projection of the combined system.

Suppose that one is studying a differential equation for which the Wahlquist-Estabrook approach yields a prolongation algebra. For each Lie group with Lie algebra arising as a homomorphic image of this prolongation algebra, there is an easy way to construct a Wahlquist-Estabrook prolongation admitting a full internal symmetry group. The method is described in the next proposition and is used in Section 6.1 to construct a prolongation of the KdV equation.

Proposition 4.5 *Let $\Delta[u] = 0$ denote a system of n-th order differential equations on $M \subseteq X \times U$ where $X = \mathbb{R}^p$ and $U = \mathbb{R}^q$ have coordinates $x = (x^1, \ldots, x^p)$ and $u = (u^1, \ldots, u^q)$ respectively. Suppose that Δ admits a Wahlquist-Estabrook prolongation described by*

$$y_i^a = \sum_{\mu=1}^{s} \sigma_i^\mu(x, u^{(n-1)}) X_\mu^a(y), \quad i = 1, \ldots, p, \quad a = 1, \ldots, r,$$

whenever the vector fields $\{X_\mu = \sum_{a=1}^r X_\mu^a(y)\partial_{y^a} : \mu = 1, \ldots, s\}$ generate a Lie algebra isomorphic to $\mathfrak{g}$, for some fixed Lie algebra $\mathfrak{g}$. Let G be any Lie group with Lie algebra $\mathfrak{g}_L$ (respectively $\mathfrak{g}_R$) of left-invariant (respectively right-invariant) vector fields such that $\mathfrak{g} \cong \mathfrak{g}_L$. Then the representation $\mathfrak{g} \to \mathfrak{g}_L$ of $\mathfrak{g}$ in terms of the left-invariant vector fields on G provides a Wahlquist-Estabrook prolongation of Δ with *pseudopotential space G. This prolongation has full internal symmetry group with every element of $\mathfrak{g}_R$ being an internal symmetry generator.*

PROOF: The observation that $[\mathfrak{g}_L, \mathfrak{g}_R] = 0$ for every Lie group G [47] proves that $\mathfrak{g}_R$ is a subalgebra of the internal symmetry algebra of this prolongation. Let f be a smooth function on $M \times G$ such that $\mathbf{v}(f) = 0$ for all $\mathbf{v} \in \mathfrak{g}_R$. Since the local coordinate expressions for these special internal symmetry generators are independent of x and u, one can treat f as a smooth function on G, parametrized by x and u. Then $\mathbf{v}(f) = 0$ for all $\mathbf{v} \in \mathfrak{g}_R$ implies that f is invariant under all right translations. Hence, f must be constant on G and the full internal symmetry group property is proved. □

4.2 Connections

Soon after the original papers on Wahlquist-Estabrook prolongations appeared in the literature, it was realized that these structures can be interpreted as connections on trivial fibre bundles. Pioneering workers in this field were Crampin [16] and Hermann [41] who showed that to each solution of the KdV and sine-Gordon equations, $0 = u_t + u_{xxx} + 12uu_x$ and $u_{xt} = \sin u$ respectively, there corresponds a set of one-forms $\{\tau^1, \tau^2, \tau^3\}$ on $\mathbb{R}^2$, the space of independent variables for these equations, satisfying the equations

$$\begin{aligned} d\tau^1 &= \tau^2 \wedge \tau^3, \\ d\tau^2 &= 2\tau^1 \wedge \tau^2, \\ d\tau^3 &= -2\tau^1 \wedge \tau^3. \end{aligned}$$

Since these are the same as the Maurer-Cartan equations satisfied by the right-invariant one-forms on the Lie group $SL(2, \mathbb{R})$, it follows that with each solution to the differential equation in question there can be associated a two-dimensional submanifold of $SL(2, \mathbb{R})$ on which the right-invariant one-forms vanish. Crampin interpreted the $\mathfrak{sl}(2, \mathbb{R})$-valued one-form

$$\Theta = \begin{pmatrix} \tau^1 & \tau^2 \\ \tau^3 & -\tau^1 \end{pmatrix}$$

as defining a connection on a principal $SL(2, \mathbb{R})$-bundle over $\mathbb{R}^2$, with the appropriate equation, either the KdV or sine-Gordon equation, reflecting the fact that this connection is flat. Later work by Shadwick [86] described the Wahlquist-Estabrook prolongation of the KdV equation in terms of flat connections on vector bundles associated with principal G-bundles, where the Lie algebra of G is a homomorphic image of the prolongation algebra of the KdV equation constructed by Wahlquist and Estabrook [92].

This section examines further, from the point of view of HC-projections, the ways in which connections arise from Wahlquist-Estabrook prolongations. The starting point involves associating a flat connection on a principal fibre bundle with every solution of an HC-projection of a differential equation.

Let $\Delta[u] = 0$ denote a system of n-th order differential equations on $M \subseteq X \times U$ and let $\Phi : N \to M^{(n)}$ be a solution of the G-induced HC-projected problem associated with Δ. Here G is an r-dimensional symmetry group of Δ which will be assumed to act freely on M in such a way that $\Phi(N) \to \Phi(N)/\mathrm{pr}^{(n)}G$ is a principal fibre bundle. Let $\{N_\gamma : \gamma \in \Gamma\}$ be the foliation of N such that each $\Phi : N_\gamma \to M^{(n)}$ is a solution of Δ.

One introduces a connection on this principal $\mathrm{pr}^{(n)}G$-bundle by defining the horizontal subspaces to be the tangent spaces to the leaves of the foliation. From the second part of Proposition 3.4, to each $g \in G$ and $\gamma \in \Gamma$ there corresponds $g(\gamma) \in \Gamma$ such that

$$\mathrm{pr}^{(n)}g \cdot \Phi(N_\gamma) \subseteq \Phi(N_{g(\gamma)})$$

so that the action of the structure group $\mathrm{pr}^{(n)}G$ on $\Phi(N)$ preserves the horizontal subspaces of the connection. A further assumption is required for such connections to exist — the action of $\mathrm{pr}^{(n)}G$ will be assumed to be transverse to the foliation of $\Phi(N)$. This means that at each point $x \in \Phi(N)$ every tangent vector $\mathbf{v}_x \in T_x\Phi(N)$ can be expressed uniquely as the sum of two vectors, tangent to the leaf through x and the $\mathrm{pr}^{(n)}G$-orbit through x respectively. That is,

$$T_x\Phi(N) = T_x\Phi(N_\gamma) + T_x(\mathrm{pr}^{(n)}G \cdot x) \tag{4.7}$$

as a vector space direct sum, where $\Phi(N_\gamma)$ is the leaf of the foliation which contains x and $\mathrm{pr}^{(n)}G \cdot x$ denotes the orbit passing through x.

The following restatement of the transversality requirement was used in Section 4.1 to associate a Wahlquist-Estabrook prolongation with an arbitrary HC-projection and will be useful when constructing the connection one-form.

Lemma 4.6 *Let $\Phi : N \to M^{(n)}$ be a solution of the G-induced HC-projected problem associated with Δ. Suppose that $\mathrm{pr}^{(n)}G$ acts freely on $\Phi(N)$ with infinitesimal generators $\{\mathbf{v}_a : a = 1, \ldots, r\}$ and that the foliation of $\Phi(N)$ is determined by the module of one-forms generated by $\{\alpha^a : a = 1, \ldots, r\}$. Then the action of $\mathrm{pr}^{(n)}G$ is transverse to the foliation if and only if the matrix $[\alpha^a(\mathbf{v}_b)]$ is invertible on $\Phi(N)$.*

PROOF: Since $\Phi(N)$ is $\mathrm{pr}^{(n)}G$-invariant and the action of $\mathrm{pr}^{(n)}G$ is free it follows that equation (4.7) is equivalent to the condition

$$T_x\Phi(N_\gamma) \cap T_x(\mathrm{pr}^{(n)}G \cdot x) = 0$$

which must hold for all $x \in \Phi(N)$. The tangent vector $\mathbf{v}_x$ belongs to the left hand side of this expression if and only if $\alpha_x^a(\mathbf{v}_x) = 0$ for all $a = 1, \ldots, r$ and

$$\mathbf{v}_x = \sum_{a=1}^{r} k^a(\mathbf{v}_a)_x$$

for some constants k^a. Thus, the group action is transverse to the foliation if and only if the only solution to the system of equations

$$0 = \alpha_x^a\left(\sum_{b=1}^{r} k^b(\mathbf{v}_b)_x\right) = \sum_{b=1}^{r} k^b(\alpha^a(\mathbf{v}_b))(x), \quad a = 1, \ldots, r,$$

is trivial. Equivalently, $[\alpha^a(\mathbf{v}_b)]$ is invertible for all $x \in \Phi(N)$. □

Of course, one can always restrict the domain of a solution $\Phi : N \to M^{(n)}$ to the open, connected subset of N on which $\det(\alpha^a(\mathbf{v}_b)) > 0$, ensuring that the action of $\mathrm{pr}^{(n)}G$ is transverse to the foliation of the corresponding restricted solution of the HC-projected problem.

Continuing with the notation of Lemma 4.6, let $\mathfrak{g}$ be the Lie algebra of G with basis $\{X_a : a = 1, \ldots, r\}$ corresponding to the infinitesimal generators $\{\mathbf{v}_a : a = 1, \ldots, r\}$ of the $\mathrm{pr}^{(n)}G$-action in the obvious way. Suppose that the Lie bracket is given by

$$[X_a, X_b] = \sum_{c=1}^{r} C_{ab}^c X_c, \quad a, b = 1, \ldots, r.$$

The one-form Θ associated with the connection defined above is

$$\Theta = \sum_{a=1}^{r} X_a \otimes \theta^a,$$

where each θ^a is a one-form on $\Phi(N)$. If Θ is to map horizontal vectors to zero then each θ^a must be contained in the module generated by $\{\alpha^a : a = 1, \ldots, r\}$, so that there exist smooth, real-valued functions $\{f^a_b : a, b = 1, \ldots, r\}$ on $\Phi(N)$ such that

$$\theta^a = \sum_{b=1}^{r} f^a_b \alpha^b, \quad a = 1, \ldots, r.$$

Furthermore, the requirement that $\Theta(\mathbf{v}) = X$ for every vertical vector $\mathbf{v}$ on $\Phi(N)$ with corresponding element $X \in \mathfrak{g}$ means that

$$X_a = \Theta(\mathbf{v}_a) = \sum_{b=1}^{r} X_b \otimes \theta^b(\mathbf{v}_a), \quad a = 1, \ldots, r,$$

implying that

$$\delta^b_a = \theta^b(\mathbf{v}_a) = \sum_{c=1}^{r} f^b_c \alpha^c(\mathbf{v}_a), \quad a, b = 1, \ldots, r. \tag{4.8}$$

By Lemma 4.6 the system of equations (4.8) has a unique solution, so that the connection one-form Θ is well-defined.

Connections derived from a foliation of a principal fibre bundle are always flat, as the Lie bracket of two horizontal vectors will also be horizontal, so that the curvature two-form will vanish [49]. Consequently,

$$d\theta^a + \sum_{1 \le b < c \le r} C^a_{bc} \theta^b \wedge \theta^c = 0, \quad a = 1, \ldots, r, \tag{4.9}$$

on $\Phi(N)$.

Example 4.7 Parametrizing solutions of the three-extended problem associated with the heat equation by (x, t, v, p, q), with $p = v_x$ and $q = v_t$, leads to a three-extended equation associated with the heat equation of the form

$$\begin{aligned} m_t &= m_{xx} + 2pm_{xv} + 2qm_{xp} + 2mm_{xq} \\ &\quad + p^2 m_{vv} + 2pqm_{vp} + 2pmm_{vq} + q^2 m_{pp} + 2qmm_{pq} + m^2 m_{qq}, \end{aligned} \tag{4.10}$$

where $v_{xt} = m(x, t, v, p, q)$. Foliating solutions of the three-extended problem using the forms

$$\begin{aligned} \theta &= dv - pdx - qdt, \\ \theta_x &= dp - qdx - mdt, \\ \theta_t &= dq - mdx - (m_x + pm_v + qm_p + mm_q)dt, \end{aligned}$$

results in three-parameter families of solutions to the heat equation.

For each $a \in \{1,2,3\}$ let $\mathbf{w}_a$ denote the appropriate prolongation of the symmetry generator $\mathbf{v}_{2a-1}$ featured in Example 3.8. In this way, a three-dimensional symmetry group of equation (4.10) is obtained from a certain three-dimensional symmetry group of the heat equation. Looking for the corresponding group-invariant solutions of equation (4.10) leads to the *Ansatz*

$$m(x,t,v,p,q) = v \cdot r(y,t) - 2v^{-2}p^3 + 3v^{-1}pq$$

where $y = v^{-1}q - v^{-2}p^2$. The reduced equation for the function $r(y,t)$ is

$$r_t = r^2 r_{yy} - 2y^2 r_y + 6yr \tag{4.11}$$

which, by the preceding theory, is a third order HC-projection of the heat equation. This equation has also appeared in work by Sokolov, Svinolupov and Wolf [87]. The components $\{\theta^1, \theta^2, \theta^3\}$ of the connection form satisfy $\theta^a(\mathbf{w}_b) = \delta^a_b$ for all $a, b = 1, 2, 3$ and are given by

$$\begin{aligned}
\theta^1 &= \left(\frac{2p^2 - 2tq^2 - vq + 2tpm}{3vpq - 2p^3 - v^2m}\right)\theta \\
&\quad - 2\left(\frac{vp - tpq + tvm}{3vpq - 2p^3 - v^2m}\right)\theta_x + \left(\frac{v^2 - 2tp^2 + 2tvq}{3vpq - 2p^3 - v^2m}\right)\theta_t, \\
\theta^2 &= \left(\frac{2pq + xq^2 - vm - xpm}{3vpq - 2p^3 - v^2m}\right)\theta \\
&\quad - \left(\frac{xpq + 2p^2 - xvm}{3vpq - 2p^3 - v^2m}\right)\theta_x + \left(\frac{xp^2 + vp - xvq}{3vpq - 2p^3 - v^2m}\right)\theta_t, \\
\theta^3 &= -\left(\frac{q^2 - pm}{3vpq - 2p^3 - v^2m}\right)\theta \\
&\quad - \left(\frac{pq - vm}{3vpq - 2p^3 - v^2m}\right)\theta_x + \left(\frac{vq - p^2}{3vpq - 2p^3 - v^2m}\right)\theta_t.
\end{aligned}$$

If $\Phi : N \to M^{(2)}$ is a solution of the three-extended problem invariant under the symmetry group generated by $\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$ then

$$d\theta^1 = d\theta^2 - \theta^1 \wedge \theta^3 = d\theta^3 = 0 \tag{4.12}$$

on $\Phi(N)$.

The change of coordinates $(x, t, v, p, q, m) \mapsto (x, t, v, p, y, r)$ was made in deriving the projected equation (4.11). In terms of the new coordinates

$$\begin{aligned}
\theta^1 &= d(x + 2tv^{-1}p) + \tau^1, \\
\theta^2 &= d(\log v + tv^{-2}p^2) - (x + 2tv^{-1}p)d(v^{-1}p) \\
&\quad + (v^{-1}p)\tau^1 + \tau^2 - (x + 2tv^{-1}p)\tau^3, \\
\theta^3 &= d(v^{-1}p) + \tau^3,
\end{aligned}$$

where

$$\begin{aligned}
\tau^1 &= -\frac{1 + 2yt}{r}dy + \frac{(1 + 2yt)(rr_y + 2y^2) - 2tr^2}{r}dt, \\
\tau^2 &= -ydt, \\
\tau^3 &= -\frac{y}{r}dy + \frac{y(rr_y + 2y^2) - r^2}{r}dt.
\end{aligned}$$

An immediate consequence of equations (4.12) is that

$$d\tau^1 = d\tau^2 + \tau^1 \wedge \tau^3 = d\tau^3 = 0 \qquad (4.13)$$

on solutions to equation (4.11).

The reader may be suspicious of the form in which the components of the connection form have been written in order to obtain the one-forms $\{\tau^1, \tau^2, \tau^3\}$. That this expansion is a natural one will become clear later in this section, where a rigorous construction of such one-forms will be presented. □

Special systems of equations involving differential forms, such as equations (4.13), have been studied by Estabrook [22], who called them *invariant Pfaffian systems.* The equations in question involve an anholonomic basis of one-forms, $\{\tau^1, \tau^2, \tau^3\}$ in the case of equations (4.13), and expressions involving only exterior derivatives and wedge products of these one forms, with just constant coefficients appearing. Estabrook, and later Hoenselaers [42], [43], investigated the relationship between these systems and the prolongation algebras occurring in Wahlquist-Estabrook prolongation theory. Harrison [38], calling them *constant coefficient ideals*, used such systems to rederive auto-Bäcklund transformations for the Ernst equation.

It must be emphasized that two constant coefficient ideals appear in Example 4.7. The ideal generated by $\{\tau^1, \tau^2, \tau^3\}$ is the analogue of the systems studied by Estabrook. It appears as a set of one-forms defined on the base manifold of the principal fibre bundle — in other words, it describes the projected system, equation (4.11). The other constant coefficient ideal, generated by $\{\theta^1, \theta^2, \theta^3\}$, is a set of one-forms defined on the bundle space of the principal fibre bundle. Such systems always arise from HC-projections and reflect, via equations (4.9), the fact that the connection constructed earlier is necessarily flat. The guaranteed existence of constant coefficient ideals defined on the base space will be shown later in this section.

Equations (4.9) have several applications. Perhaps the most important one, the decomposition of HC-projections, is treated later in this chapter. Another useful application involves using the connection form to simplify the construction of HC-projected equations. This method does not require the construction of extended equations and allows one to go directly from the original differential equation to the projected system. It is demonstrated by an example.

Example 4.8 Return to the first order HC-projection of the heat equation generated by the infinitesimal symmetry generator $\mathbf{v}_3$. Recall that the one-form determining the foliation is

$$\eta = dv - w dx - (w_x + w w_v) dt,$$

so that $\eta(\mathbf{v}_3) = v$. Therefore, everywhere v is nonzero there exists a connection form Θ with sole component θ given by

$$\theta = v^{-1}\eta = d \log v - v^{-1} w dx - v^{-1}(w_x + w w_v) dt.$$

The flatness of the connection is reflected by the equation

$$\begin{aligned} 0 = d\theta &= v^{-2}(w - v w_v) dv \wedge dx \\ &\quad + v^{-2}(w_x + w w_v - v(w_{xv} + w_v^2 + w w_{vv})) dv \wedge dt \\ &\quad + v^{-1}(w_t - w_{xx} - w_x w_v - w w_{xv}) dx \wedge dt. \end{aligned}$$

From the coefficient of $dv \wedge dx$, $w - v w_v = 0$, implying that $w(x,t,v) = v \cdot u(x,t)$ for some function u. The coefficient of $dv \wedge dt$ then vanishes automatically and that of $dx \wedge dt$ leads to Burgers' equation. □

Thus, given the infinitesimal symmetries generating the HC-projection, one constructs the connection form by solving the system of equations (4.8) and substitutes the result into the zero-curvature condition, equations (4.9). Effectively, the problem of reducing the extended equation using the appropriate symmetry group is attacked via the method of side conditions as studied by Olver and Rosenau [73]. The projected equation appears as an overdetermined system equivalent to the extended equation together with side conditions corresponding to the appropriate group reduction.

This section continues with a discussion indicating how the constant coefficient ideals of Estabrook can be derived from the connection form associated with an HC-projection. In particular, it is shown that systems of equations slightly more general than those appearing in Example 4.7 can be obtained.

Each solution of the G-induced HC-projected problem associated with Δ, subject to some technical requirements, leads to a principal G-bundle on which a flat connection can be defined. Such a bundle is necessarily trivial [17]. Let $\Phi : N \rightarrow M^{(n)}$ be a solution to the HC-projected problem, so that $\Phi(N) = X \times G$ for some p-dimensional manifold X, and suppose that X and G have coordinates $x = (x^1, \ldots, x^p)$ and $y = (y^1, \ldots, y^r)$ respectively. It follows that the action of the structure group G has infinitesimal generators

$$\mathbf{v}_a = \sum_{b=1}^{r} \eta_a^b(y)\partial_{y^b}, \quad a = 1, \ldots, r, \tag{4.14}$$

where each $\mathbf{v}_a$ is a right-invariant vector field on G. From Section 4.1 and the fact that the transversality requirement needed there is satisfied, the foliation of $\Phi(N)$ into solutions of Δ is determined by one-forms

$$\omega^a = dy^a - \sum_{i=1}^{p} F_i^a(x, y)dx^i, \quad a = 1, \ldots, r,$$

for suitable smooth functions F_i^a. The G-action must preserve the module of forms generated by $\{\omega^a : a = 1, \ldots, r\}$ due to the properties of a connection on a principal fibre bundle. Since

$$\mathcal{L}_{\mathbf{v}_a}\omega^b - \sum_{c=1}^{r} \frac{\partial \eta_a^b}{\partial y^c}\omega^c = \sum_{i=1}^{p}\sum_{c=1}^{r} \left(F_i^c \frac{\partial \eta_a^b}{\partial y^c} - \eta_a^c \frac{\partial F_i^b}{\partial y^c} \right) dx^i, \quad a, b = 1, \ldots, r,$$

where $\mathcal{L}_{\mathbf{v}_a}$ denotes the Lie derivative along $\mathbf{v}_a$, it follows that

$$\sum_{c=1}^{r} \left(F_i^c \frac{\partial \eta_a^b}{\partial y^c} - \eta_a^c \frac{\partial F_i^b}{\partial y^c} \right) = 0, \quad i = 1, \ldots, p, \quad a, b = 1, \ldots, r, \tag{4.15}$$

on $X \times G$. The following result can be used to find the general solution to equations (4.15).

Lemma 4.9 *Let* $\{\mathbf{v}_a : a = 1, \ldots, r\}$ *denote a basis for the right-invariant vector fields on a connected Lie group* G. *If the vector field* $\mathbf{u}$ *on* G *satisfies*

$$[\mathbf{u}, \mathbf{v}_a] = 0, \quad a = 1, \ldots, r,$$

then $\mathbf{u}$ *must be left-invariant.*

PROOF: Let $\exp(a\mathbf{u})$ denote the one-parameter group of transformations generated by $\mathbf{u}$ and let $g \in G$ be arbitrary. Then there exist right-invariant vector fields $\mathbf{w}_1, \ldots, \mathbf{w}_k$ on G such that

$$g = \exp(\mathbf{w}_1) \cdot \exp(\mathbf{w}_2) \cdots \exp(\mathbf{w}_k).$$

Since $[\mathbf{u}, \mathbf{w}_j] = 0$ for all $j = 1, \ldots, k$, it follows that for every $h \in G$

$$\begin{aligned} \exp(a\mathbf{u})(\exp(\mathbf{w}_j) \cdot h) &= \exp(a\mathbf{u})(\exp(\mathbf{w}_j)(h)) \\ &= \exp(\mathbf{w}_j)(\exp(a\mathbf{u})(h)) \\ &= \exp(\mathbf{w}_j) \cdot (\exp(a\mathbf{u})(h)) \end{aligned}$$

as the one-parameter subgroup $\exp(a\mathbf{w}_j)$ acts on G via multiplication on the left by $\exp(a\mathbf{w}_j) \equiv \exp(a\mathbf{w}_j)(e)$. Thus

$$\begin{aligned} \exp(a\mathbf{u})(g) &= \exp(a\mathbf{u})(\exp(\mathbf{w}_1) \cdot \exp(\mathbf{w}_2) \cdots \exp(\mathbf{w}_k) \cdot e) \\ &= \exp(\mathbf{w}_1) \cdot \exp(\mathbf{w}_2) \cdots \exp(\mathbf{w}_k) \cdot (\exp(a\mathbf{u})(e)) \\ &= g \cdot (\exp(a\mathbf{u})(e)), \end{aligned}$$

so that $\exp(a\mathbf{u})$ acts on G via right multiplication by $\exp(a\mathbf{u}) \equiv \exp(a\mathbf{u})(e)$. Consequently, $\mathbf{u}$ must be a left-invariant vector field. □

Equations (4.15) and Lemma 4.9 enable the y-dependence of the functions F_i^a to be completely determined. The vector fields

$$\mathbf{u}_i = \sum_{a=1}^{r} F_i^a \partial_{y^a}, \quad i = 1, \ldots, p,$$

must commute with every right-invariant vector field $\mathbf{v}_a$. That is, for each fixed $x \in X$, $\mathbf{u}_i|_x$ is a left-invariant vector field on G, whence

$$\mathbf{u}_i|_x = \sum_{a=1}^{r} \sigma_i^a \mathbf{v}_a^*, \quad i = 1, \ldots, p,$$

where σ_i^a are constants and the vector fields

$$\mathbf{v}_a^* = \sum_{b=1}^{r} \xi_a^b(y) \partial_{y^b}, \quad a = 1, \ldots, r,$$

are the left-invariant vector fields on G such that $\mathbf{v}_a^*|_e = \mathbf{v}_a|_e$. For later use, notice that

$$[\mathbf{v}_a^*, \mathbf{v}_b^*] = -\sum_{c=1}^{r} C_{ab}^c \mathbf{v}_c^*, \quad a, b = 1, \ldots, r, \tag{4.16}$$

where C_{ab}^c are the structure constants for the algebra of right-invariant vector fields:

$$[\mathbf{v}_a, \mathbf{v}_b] = \sum_{c=1}^{r} C_{ab}^c \mathbf{v}_c, \quad a, b = 1, \ldots, r,$$

(see Exercise 1.33 of [72]). Therefore,

$$\mathbf{u}_i = \sum_{a=1}^{r} \sigma_i^a(x) \mathbf{v}_a^*, \quad i = 1, \ldots, p,$$

for suitable smooth functions σ_i^a and

$$F_i^a(x, y) = \sum_{b=1}^{r} \sigma_i^b(x) \xi_b^a(y), \quad i = 1, \ldots, p, \ a = 1, \ldots, r. \tag{4.17}$$

The one-forms determining the foliation of $\Phi(N)$ are thus

$$\omega^a = dy^a - \sum_{i=1}^{p} F_i^a dx^i = dy^a - \sum_{b=1}^{r} \xi_b^a(y) \beta^b, \quad a = 1, \ldots, r, \tag{4.18}$$

where

$$\beta^a = \sum_{i=1}^{p} \sigma_i^a(x) dx^i, \quad a = 1, \ldots, r,$$

are suitable one-forms on X. Closure under exterior differentiation of the ideal generated by $\{\omega^a : a = 1, \ldots, r\}$ implies that

$$\begin{aligned}
0 &= d\omega^a + \sum_{b,c=1}^{r} \frac{\partial \xi_b^a}{\partial y^c} \omega^c \wedge \beta^b \\
&= -\sum_{b=1}^{r} \xi_b^a d\beta^b + \sum_{b,c,e=1}^{r} \xi_e^c \frac{\partial \xi_b^a}{\partial y^c} \beta^b \wedge \beta^e \\
&= -\sum_{b=1}^{r} \xi_b^a d\beta^b - \sum_{1 \le b < e \le r} \sum_{c=1}^{r} \left(\xi_b^c \frac{\partial \xi_e^a}{\partial y^c} - \xi_e^c \frac{\partial \xi_b^a}{\partial y^c} \right) \beta^b \wedge \beta^e
\end{aligned}$$

and, using equations (4.16), it follows that

$$\sum_{b=1}^{r} \xi_b^a d\beta^b = \sum_{c=1}^{r} \xi_c^a \sum_{1\le b<e\le r} C_{be}^c \beta^b \wedge \beta^e, \quad a = 1,\dots,r.$$

Regularity of the group action implies that the matrix $[\xi_b^a]$ is invertible, which allows one to write these equations in the form

$$d\beta^a = \sum_{1\le b<c\le r} C_{bc}^a \beta^b \wedge \beta^c, \quad a = 1,\dots,r. \tag{4.19}$$

That is, $\{\beta^a : a = 1,\dots,r\}$ provides one constant coefficient ideal for the system under consideration.

However, there is no guarantee that the one-forms $\{\beta^a : a = 1,\dots,r\}$ are linearly independent over $\mathbb{R}$. Let $\{\tau^\mu : \mu = 1,\dots,s\}$ be a basis for these one-forms, so that $s \le r$ and there exist constants $\{N_\mu^a : a = 1,\dots,r,\ \ \mu = 1,\dots,s\}$ such that

$$\beta^a = \sum_{\mu=1}^{s} N_\mu^a \tau^\mu, \quad a = 1,\dots,r.$$

Thus the forms determining the foliation are

$$\omega^a = dy^a - \sum_{b=1}^{r} \xi_b^a \left(\sum_{\mu=1}^{s} N_\mu^b \tau^\mu\right) = dy^a - \sum_{\mu=1}^{s}\left(\sum_{b=1}^{r} N_\mu^b \xi_b^a\right)\tau^\mu, \quad a = 1,\dots,r,$$

and equations (4.19) then yield the system of r equations

$$\sum_{\mu=1}^{s} N_\mu^a d\tau^\mu = \sum_{1\le b<c\le r} C_{bc}^a \left(\sum_{\mu=1}^{s} N_\mu^b \tau^\mu\right) \wedge \left(\sum_{\mu=1}^{s} N_\mu^c \tau^\mu\right), \quad a = 1,\dots,r,$$

involving s one-forms $\{\tau^\mu : \mu = 1,\dots,s\}$. These equations are the analogues of the constant coefficient ideals of Estabrook. The possibility of the system containing more than s independent equations allows equations such as

$$0 = \tau^1 \wedge \tau^3 + \tau^2 \wedge \tau^4,$$

which are typical of those occurring in the work of Estabrook and other authors.

Example 4.10 The promised justification of the final part of Example 4.7 is now presented. It illustrates much of the preceding discussion as well. In order to obtain a trivialization of the principal fibre bundle appearing in that example, it is necessary

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to make a coordinate change in addition to $(x,t,v,p,q,m) \mapsto (x,t,v,p,y,r)$, which arose during the group reduction process. One such change of coordinates is given by $(x,t,v,p,y,r) \mapsto (y,t,r,\alpha,\beta,\gamma)$ where

$$\alpha = x + 2tv^{-1}p, \quad \beta = \log v + tv^{-2}p^2, \quad \gamma = v^{-1}p,$$

and leads to infinitesimal generators of the group action of the form

$$\mathbf{w}_1 = \partial_\alpha, \quad \mathbf{w}_2 = \partial_\beta, \quad \mathbf{w}_3 = \alpha\partial_\beta + \partial_\gamma.$$

Notice the agreement with equations (4.14). Referring back to Example 4.7, one finds that the components of the connection form are

$$\begin{aligned} \theta^1 &= d\alpha + \tau^1, \\ \theta^2 &= d\beta - \alpha d\gamma + \gamma\tau^1 + \tau^2 - \alpha\tau^3, \\ \theta^3 &= d\gamma + \tau^3. \end{aligned}$$

These forms yield foliation-determining one-forms which separate into the form given by equations (4.17). Furthermore, coefficients of the one-forms $\{\tau^1, \tau^2, \tau^3\}$ yield the vector fields

$$\mathbf{w}_1^* = \partial_\alpha + \gamma\partial_\beta, \quad \mathbf{w}_2^* = \partial_\beta, \quad \mathbf{w}_3^* = \partial_\gamma,$$

satisfying the commutator relations, equations (4.16). Observe that $\{\mathbf{w}_a : a = 1,2,3\}$ and $\{\mathbf{w}_a^* : a = 1,2,3\}$ are indeed right- and left-invariant vector fields, respectively, on a Lie group. The group in question is described by the multiplication operation

$$(\alpha,\beta,\gamma)\cdot(\bar{\alpha},\bar{\beta},\bar{\gamma}) = (\alpha+\bar{\alpha}, \beta+\bar{\beta}+\gamma\bar{\alpha}, \gamma+\bar{\gamma})$$

defined on $\mathbb{R}^3$. □

Finally, notice that the construction before Example 4.10 can be generalized to an arbitrary solution of the HC-projected equation and yields Wahlquist-Estabrook prolongation equations

$$y_i^a = F_i^a(x, u^{(n-1)}, y) = \sum_{b=1}^{r} \sigma_i^b(x, u^{(n-1)})\xi_b^a(y), \quad i = 1, \ldots, p, \quad a = 1, \ldots, r.$$

Thus, subject to the technical considerations needed for a flat connection to exist, every nondegenerate Wahlquist-Estabrook prolongation with full internal symmetry group can be described locally using the method of Proposition 4.5.

4.3 Decomposition of HC-projections

Suppose that one is given a solution to a particular HC-projected problem associated with some differential equation. One knows that this solution foliates into solutions of the original differential equation, one even knows the one-forms determining that foliation — but the calculation of the leaves can still be very difficult. Fortunately, the connection one-form introduced in Section 4.2 can sometimes greatly simplify the construction. The key to such simplifications is the algebraic structure of the symmetry group inducing the HC-projection.

An important class of HC-projections, and the simplest to study, are those generated by one-dimensional symmetry groups G which therefore have Abelian Lie algebras. Suppose that $\Phi : N \to M^{(n)}$ is a solution of the G-induced HC-projected problem associated with a system of n-th order differential equations Δ on $M \subseteq X \times U$. Assuming that the conditions of Section 4.2 are satisfied, there exists a connection form on the principal $\mathrm{pr}^{(n)}G$-bundle $\Phi(N) \to \Phi(N)/\mathrm{pr}^{(n)}G$ whose sole component θ^1 satisfies $d\theta^1 = 0$. The foliation of $\Phi(N)$ into solutions of Δ thus takes a particularly simple form. Locally, there exists a smooth, real-valued function f on $\Phi(N)$ such that $\theta^1 = df$. The solutions of Δ are described by the equations $f = c$ with parameter $c \in \mathbb{R}$. Therefore, a solution of $\Pi_G(\Delta)$ lifts to a one-parameter family of solutions of Δ through solving a first order system of equations by quadrature.

Suppose that an explicit parametrization of solutions to the one-extended problem associated with Δ has been made, so that now $\Pi_G(\Delta)$ represents a differential equation. The discussion above, when combined with the results of Sections 4.1 and 4.2, shows that one can recover Δ from $\Pi_G(\Delta)$ by constructing a Wahlquist-Estabrook prolongation involving a single potential. In particular, it would not be necessary to introduce a genuine pseudopotential. Let $\Phi : N \to M^{(n)}$ be the solution of the one-extended problem associated with Δ which corresponds to a particular solution of $\Pi_G(\Delta)$ and choose coordinates $(x^1, \ldots, x^p, y)$ on $\Phi(N)$ such that the infinitesimal generator of the $\mathrm{pr}^{(n)}G$-action takes the form $\mathbf{v} = \partial_y$. Such a choice is always possible because of the assumptions inherited from Section 4.2. In particular, Lemma 4.6 shows that, due to the transversality of the group action, $\mathbf{v}$ cannot vanish anywhere on $\Phi(N)$. Equation (4.8) and the equation $d\theta^1 = 0$ imply that the component of the connection one-form is

$$\theta^1 = dy + \sum_{i=1}^{p} g_i dx^i,$$

where the smooth functions g_i depend on $x^1, \ldots, x^p$ only. From the equation $\theta^1 = df$ it follows that $f = y + \beta(x^1, \ldots, x^p)$, where the function β satisfies

$$\frac{\partial \beta}{\partial x^i} = g_i(x^1, \ldots, x^p), \quad i = 1, \ldots, p.$$

Due to the form of these equations, β will be a potential of $\Pi_G(\Delta)$.

This appears to contradict the results of Example 4.1 where Wahlquist-Estabrook prolongations were constructed from the first order HC-projections of the heat equation featured in Example 3.8. For instance, in lifting solutions of equation (3.12) up to solutions of the heat equation one must solve the first order system

$$t_y = \frac{-u(1+4t^2)}{u_y + uu_z + y^2 z - bz}, \quad t_z = \frac{1+4t^2}{u_y + uu_z + y^2 z - bz},$$

for $t(y,z)$. Clearly t is a pseudopotential, and not a potential, of equation (3.12). However, the change of coordinate $s = \frac{1}{2}\arctan 2t$ converts this system to one involving a potential s of equation (3.12). Similar coordinate changes convert the other first order systems in Example 4.1 into potential form.

The foliation of a solution to an HC-projected problem into solutions of the parent differential equation is greatly simplified by knowledge of the connection form associated with the HC-projection. This is not just when the symmetry group involved is one-dimensional, but is also true for arbitrary HC-projections. The following example demonstrates this use of the connection form and motivates much of the current section.

Example 4.11 Return to Example 4.7 and suppose that one wishes to lift a solution $r(y,t)$ of the projected equation (4.11) to yield a three-parameter family of solutions to the heat equation. Equivalently, one must foliate the corresponding solution $\Phi : N \to M^{(2)}$ of the three-extended problem associated with the heat equation. In the earlier example it was shown that the foliation was generated by the three components $\{\theta^1, \theta^2, \theta^3\}$ of the connection form. These forms satisfy

$$d\theta^1 = d\theta^2 - \theta^1 \wedge \theta^3 = d\theta^3 = 0$$

on $\Phi(N)$.

Therefore, locally there exists a real-valued function $f = f(x,t,v,p,y)$ on this submanifold such that $\theta^3 = df$. It follows that $f = v^{-1}p + z(y,t)$, where the function z must satisfy the equations

$$z_y = -\frac{y}{r}, \quad z_t = \frac{y(rr_y + 2y^2) - r^2}{r}. \tag{4.20}$$

Notice that z is a potential of equation (4.11). The five-dimensional solution manifold $\Phi(N)$ of the three-extended problem foliates into four-dimensional leaves given by $f = c_1$, for a parameter $c_1 \in \mathbb{R}$. Since z is a potential it is only defined up to addition by an arbitrary constant, so, without loss of generality, the leaves can be described by $f = 0$. Thus each leaf is given by $p = -v \cdot z(y,t)$.

Now attention is restricted to just one of these leaves. Because $\theta^3 = 0$ on that leaf, it follows that $d\theta^1 = d\theta^2 = 0$ and locally there exists a real-valued function $g = g(x,t,v,y)$ on the leaf such that $\theta^1 = dg$. A simple calculation shows that $g = x - 2t \cdot z(y,t) + m(y,t)$, where m must satisfy

$$m_y = -\frac{1+2yt}{r}, \quad m_t = \frac{(1+2yt)(rr_y + 2y^2) - 2tr^2}{r}, \tag{4.21}$$

indicating that m is also a potential of equation (4.11). It is now possible to foliate the four-dimensional leaf, given by $f = 0$, into three-dimensional leaves by putting $g = c_2$ for some parameter $c_2 \in \mathbb{R}$. As before, one can take $c_2 = 0$ without loss of generality, so that the three-dimensional leaves are described by $x = 2t \cdot z(y,t) - m(y,t)$ and $p = -v \cdot z(y,t)$.

Since $d\theta^2 = 0$ on the leaf $f = g = 0$, locally there exists a real-valued function $h = h(t,v,y)$ such that $\theta^2 = dh$. This time $h = \log v + t \cdot (z(y,t))^2 + n(y,t)$, where n must satisfy

$$n_y = \frac{z(1+2yt)}{r}, \quad n_t = \frac{z\,(2tr^2 - (1+2yt)(rr_y + 2y^2))}{r} - y, \tag{4.22}$$

and n is a potential of the system of equations (4.11) and (4.20). Each three-dimensional leaf now foliates into two-dimensional leaves after imposing the constraint $h = c_3$ with $c_3 \in \mathbb{R}$ a parameter. Once more, $c_3 = 0$ without loss of generality and, therefore, the solution $r(y,t)$ of equation (4.11) lifts to a three-parameter family of solutions to the heat equation, described parametrically by

$$\begin{aligned} x &= 2t \cdot z(y,t) - m(y,t), \\ v &= \exp(-t \cdot (z(y,t))^2 - n(y,t)). \end{aligned}$$

Eliminating y between these two equations yields solutions $v = v(x,t)$ in the usual form. Therefore, to lift a solution of equation (4.11) to solutions of the heat equation, one need only solve three first order systems by quadrature. It will be shown later in the current section that this is precisely because the symmetry algebra spanned by $\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$ of Example 4.7 is solvable. □

This example shows how the process of foliating solutions of an HC-projected problem can sometimes be replaced by a sequence of simpler foliations. Such a simplification is possible whenever the symmetry group generating the HC-projection has a subgroup. The following proposition states this result rigorously. Its proof indicates the utility of the connection form in the simplification process.

Proposition 4.12 *Let Δ denote a system of n-th order differential equations on $M \subseteq X \times U$ with a symmetry group G which has a subgroup H. Suppose that $\Phi : N \to M^{(n)}$ is a solution of the G-induced HC-projected problem associated with Δ such that $\Phi(N) \to \Phi(N)/\mathrm{pr}^{(n)}G$ is a principal $\mathrm{pr}^{(n)}G$-bundle and $\mathrm{pr}^{(n)}G$ acts transversally to the connection described in Section 4.2. Then*

1. *N admits a codimension $(\dim G - \dim H)$ foliation $\{N_\alpha : \alpha \in A\}$ such that each $\Phi : N_\alpha \to M^{(n)}$ is a solution of the H-induced HC-projected problem associated with Δ.*

2. *Each N_α admits a codimension $\dim H$ foliation $\{N_{\alpha,\beta} : \beta \in B\}$ such that each $\Phi : N_{\alpha,\beta} \to M^{(n)}$ is a solution of Δ.*

3. *The induced foliation $\{N_{\alpha,\beta} : (\alpha,\beta) \in A \times B\}$ of N coincides with the usual foliation of N into solutions of Δ.*

PROOF: Suppose that $\{\mathbf{v}_1, \ldots, \mathbf{v}_r\}$ is a basis of infinitesimal generators for the action of $\mathrm{pr}^{(n)}G$ on $M^{(n)}$, with $\{\mathbf{v}_1, \ldots, \mathbf{v}_s\}$ generating the action of the subgroup $\mathrm{pr}^{(n)}H$. If the commutators of these vector fields are given by

$$[\mathbf{v}_a, \mathbf{v}_b] = \sum_{c=1}^{r} C^c_{ab}\mathbf{v}_c, \quad a, b = 1, \ldots, r,$$

then, from Section 4.2, the restriction of the contact module $\Omega^{(n)}$ to $\Phi(N)$ is spanned by one-forms $\{\theta^1, \ldots, \theta^r\}$ which satisfy

$$0 = d\theta^c + \sum_{1 \le a < b \le r} C^c_{ab}\theta^a \wedge \theta^b, \quad c = 1, \ldots, r.$$

Since H is a subgroup of G it follows that $C^c_{ab} = 0$ whenever $a, b \leq s$ and $c \geq s+1$, implying that $\{\theta^{s+1}, \ldots, \theta^r\}$ generate a closed ideal of forms in their own right. Let $\{N_\alpha : \alpha \in A\}$ denote the corresponding foliation of N.

Now fix $\alpha \in A$. By construction, the contact module $\Omega^{(n)}$ is generated by $\{\theta^1, \ldots, \theta^s\}$ when restricted to $\Phi(N_\alpha)$. Since $\theta^{s+1} = \cdots = \theta^r = 0$ on $\Phi(N_\alpha)$, the ideal generated by $\{\theta^1, \ldots, \theta^s\}$ is closed under exterior differentiation with

$$0 = d\theta^c + \sum_{1 \leq a < b \leq s} C^c_{ab} \theta^a \wedge \theta^b, \quad c = 1, \ldots, s.$$

If $X = \mathbb{R}^p$ then clearly $\Phi(N_\alpha)$ is $(p+s)$-dimensional and contained in the subvariety $\mathcal{S}_\Delta$ of $M^{(n)}$ representing the differential equation Δ, so that $\Phi : N_\alpha \to M^{(n)}$ is a solution of the s-extended problem associated with Δ. Furthermore, since each of the vector fields $\mathbf{v}_1, \ldots, \mathbf{v}_s$ is tangent to $\Phi(N_\alpha)$, it follows that $\Phi(N_\alpha)$ is locally $\mathrm{pr}^{(n)} H$-invariant and thus a solution of the H-induced HC-projected problem associated with Δ. This completes the proof of the first part of the proposition.

The remainder of the proof follows directly. Simply foliate $\Phi(N_\alpha)$ using the forms $\{\theta^1, \ldots, \theta^s\}$ and appeal to the uniqueness result of Proposition 3.2. □

Proposition 4.12 is typical of the results which can be obtained via the geometric approach to HC-projections. When H is a normal subgroup of G, and this assumption is made for the remainder of the current section, Propositions 3.11 and 4.12 suggest that the G-induced HC-projection of a differential equation Δ can be decomposed into the H-induced HC-projection of that equation, followed by the K-induced HC-projection of $\Pi_H(\Delta)$. Here K is a symmetry group of $\Pi_H(\Delta)$ which is isomorphic to G/H. However, proving the result in this manner is not entirely satisfactory because, in terms of the geometric approach, the above decomposition is meaningless — an HC-projected problem associated with another HC-projected problem has not been defined! Nevertheless, Proposition 4.12 is included due to the geometric insight it provides and because it will be useful when the Miura transformation is generalized in the next section. The complication above can be avoided by studying the decomposition of HC-projections in terms of the equivalent concept of nondegenerate Wahlquist-Estabrook prolongations with full internal symmetry groups.

Proposition 4.13 *Let Δ denote a system of n-th order differential equations on $M \subseteq X \times U$ with (Δ, Ξ) a nondegenerate Wahlquist-Estabrook prolongation to $M \times W$. Suppose that this prolongation has full internal symmetry group G with a normal subgroup H.*

1. *There exists a nondegenerate Wahlquist-Estabrook prolongation (Δ, Υ) of Δ to $M \times Y$ admitting a full internal symmetry grouop. This prolongation has internal symmetry algebra isomorphic to $\mathfrak{g}/\mathfrak{h}$.*

2. *There exists a nondegenerate Wahlquist-Estabrook prolongation $(\Delta, \Upsilon, \Lambda)$ of (Δ, Υ) to $M \times Y \times Z$ admitting a full internal symmetry group. This prolongation has internal symmetry algebra isomorphic to $\mathfrak{h}$.*

3. *Locally $W = Y \times Z$ and there exists a change of coordinates transforming (Δ, Ξ) to $(\Delta, \Upsilon, \Lambda)$.*

PROOF: Let $X = \mathbb{R}^p$ and $U = \mathbb{R}^q$ have coordinates $x = (x^1, \ldots, x^p)$ and $u = (u^1, \ldots, u^q)$ respectively and suppose that $W = \mathbb{R}^r$. As Lemma 4.3 shows that G is r-dimensional, there exists a basis $\{\mathbf{v}_a : a = 1, \ldots, r\}$ of infinitesimal generators of the G-action on $M \times W$ such that $\{\mathbf{v}_a : a = 1, \ldots, s\}$ generates the H-action. The full internal symmetry group property implies that all H-orbits are s-dimensional whence there exists a maximal, functionally independent set $\{x^1, \ldots, x^p, u^1, \ldots, u^q, y^1, \ldots, y^{r-s}\}$ of invariants of the H-action on $M \times W$. Choose $\{z^1, \ldots, z^s\}$ so that $(x, u, y, z) = (x^1, \ldots, x^p, u^1, \ldots, u^q, y^1, \ldots, y^{r-s}, z^1, \ldots, z^s)$ give local coordinates on $M \times W$. Thus,

$$\mathbf{v}_a = \begin{cases} \displaystyle\sum_{b=1}^{s} f_a^b(x, u, y, z)\partial_{z^b}, & a = 1, \ldots, s, \\ \displaystyle\sum_{c=1}^{r-s} g_a^c(x, u, y, z)\partial_{y^c} + \sum_{b=1}^{s} h_a^b(x, u, y, z)\partial_{z^b}, & a = s+1, \ldots, r, \end{cases} \tag{4.23}$$

for appropriate smooth functions. Existence of a full internal symmetry group implies that the matrices $[f_a^b]$ and $[g_a^b]$ are invertible on $M \times W$.

In terms of the coordinates (x, u, y, z) for $M \times W$, the prolongation equations will be

$$\begin{aligned} \Upsilon_i^a &= y_i^a - F_i^a(x, u^{(n-1)}, y, z) = 0, \quad i = 1, \ldots, p, \ \ a = 1, \ldots, r-s, \\ \Lambda_i^a &= z_i^a - G_i^a(x, u^{(n-1)}, y, z) = 0, \quad i = 1, \ldots, p, \ \ a = 1, \ldots, s, \end{aligned}$$

for suitable smooth functions F_i^a, G_i^a. Since the vectors $\{\mathbf{v}_a : a = 1, \ldots, s\}$ are internal symmetry generators of this prolongation of Δ, it follows that

$$0 = \mathrm{pr}^{(1)}\mathbf{v}_b(\Upsilon_i^a) = -\sum_{c=1}^{s} f_b^c \frac{\partial F_i^a}{\partial z^c}, \quad i = 1, \ldots, p, \ \ a, b = 1, \ldots, s,$$

on $\mathcal{S}_\Delta \times W$. Invertibility of $[f_a^b]$ shows that each F_i^a is independent of z so that

$$\Upsilon_i^a = y_i^a - F_i^a(x, u^{(n-1)}, y) = 0, \quad i = 1, \ldots, p, \ \ a = 1, \ldots, r-s,$$

define a Wahlquist-Estabrook prolongation (Δ, Υ) of Δ to $M \times Y$, where Y has coordinates $y = (y^1, \ldots, y^{r-s})$. Because $\mathfrak{h}$ is an ideal of $\mathfrak{g}$, and using the invertibility of $[f_a^b]$,

$$\frac{\partial g_a^c}{\partial z^b} = 0, \quad a = s+1, \ldots, r, \ \ b = 1, \ldots, s, \ \ c = 1, \ldots, r-s,$$

in equation (4.23). Thus, the vector fields

$$\mathbf{v}_a^* = \sum_{c=1}^{r-s} g_a^c(x, u, y)\partial_{y^c}, \quad a = s+1, \ldots, r,$$

on $M \times Y$ span a Lie algebra isomorphic to $\mathfrak{g}/\mathfrak{h}$. Clearly each $\mathbf{v}_a^*$ must be an internal symmetry generator of the prolongation (Δ, Υ) of Δ. This prolongation admits a full internal symmetry group due to the invertibility of $[g_a^b]$. From Lemma 4.3, the internal symmetry group of (Δ, Υ) is $(r-s)$-dimensional and therefore generated by $\{\mathbf{v}_a^* : a = s+1, \ldots, r\}$.

The remaining prolongation equations

$$\Lambda_i^a = z_i^a - G_i^a(x, u^{(n-1)}, y, z), \quad i = 1, \ldots, p, \ \ a = 1, \ldots, s,$$

define a Wahlquist-Estabrook prolongation of (Δ, Υ). The resulting system of equations $(\Delta, \Upsilon, \Lambda)$ is equivalent to (Δ, Ξ) by construction. Of the internal symmetry generators of (Δ, Ξ), only $\{\mathbf{v}_a : a = 1, \ldots, s\}$ are internal symmetry generators of $(\Delta, \Upsilon, \Lambda)$ when it is treated as a Wahlquist-Estabrook prolongation of (Δ, Υ). Existence of a full internal symmetry group follows from $[f_a^b]$ being invertible. This prolongation features an s-dimensional internal symmetry algebra, by Lemma 4.3, which must therefore be isomorphic to $\mathfrak{h}$. □

The Wahlquist-Estabrook prolongation (Δ, Υ) constructed in this proposition is an example of an *intermediate Wahlquist-Estabrook prolongation.*

Proposition 4.13 can be used to study a particular type of system introduced by Wahlquist and Estabrook [92]. Sequential prolongations will occur whenever a nondegenerate Wahlquist-Estabrook prolongation has a full internal symmetry group which is solvable.

Corollary 4.14 *Let (Δ, Ξ) denote a nondegenerate Wahlquist-Estabrook prolongation of Δ which has full internal symmetry group. If the internal symmetry group is solvable then there exist coordinates on the pseudopotential space such that this is a sequential prolongation of Δ.*

PROOF: Let $\mathfrak{g}$ be the internal symmetry algebra, with $\dim \mathfrak{g} = r$, implying that there exists a chain of subalgebras

$$\{0\} = \mathfrak{g}_0 \subset \mathfrak{g}_1 \subset \cdots \subset \mathfrak{g}_{r-1} \subset \mathfrak{g}_r = \mathfrak{g}$$

with $\dim \mathfrak{g}_k = k$ and $\mathfrak{g}_{k-1}$ an ideal in $\mathfrak{g}_k$. From Proposition 4.13, there exists a nondegenerate intermediate prolongation (Δ, Ξ^1) of Δ with full internal symmetry group. The internal symmetry algebra is isomorphic to $\mathfrak{g}_r/\mathfrak{g}_{r-1}$ and, since the corresponding HC-projection is first order, the discussion which began this section implies that (Δ, Ξ^1) can be described by a single potential of Δ. But now (Δ, Ξ) is a nondegenerate prolongation of (Δ, Ξ^1) with full internal symmetry group and internal symmetry algebra $\mathfrak{g}_{r-1}$. Applying Proposition 4.13 once more, there exists an intermediate prolongation (Δ, Ξ^1, Ξ^2) of (Δ, Ξ^1) with full internal symmetry group. The internal symmetry algebra is isomorphic to $\mathfrak{g}_{r-1}/\mathfrak{g}_{r-2}$ so that the corresponding HC-projection is first order, thus requiring only a single potential of (Δ, Ξ^1) for its description. Continuing in this way, one obtains the sequence of prolongations

$$\begin{array}{c}
(\Delta, \Xi) \cong (\Delta, \Xi^1, \ldots, \Xi^{r-1}, \Xi^r) \\
\Big\downarrow {\scriptstyle \mathfrak{g}_1/\mathfrak{g}_0} \\
(\Delta, \Xi^1, \ldots, \Xi^{r-1}) \\
\Big\downarrow {\scriptstyle \mathfrak{g}_2/\mathfrak{g}_1} \\
\vdots \\
\Big\downarrow {\scriptstyle \mathfrak{g}_{r-1}/\mathfrak{g}_{r-2}} \\
(\Delta, \Xi^1) \\
\Big\downarrow {\scriptstyle \mathfrak{g}_r/\mathfrak{g}_{r-1}} \\
\Delta
\end{array}$$

which is described by appropriate nested potentials. In the diagram above, each component prolongation is labelled by its internal symmetry algebra. □

This corollary explains the phenomenon observed in Example 4.11. There it was shown that solutions to the HC-projected equation (4.11) could be lifted to solutions of the heat equation by a sequence of three quadratures. The explanation involves interpreting the heat equation as a Wahlquist-Estabrook prolongation of equation (4.11), a prolongation with solvable full internal symmetry group. Appropriate nested potentials are described by equations (4.20) to (4.22).

The promised decomposition of HC-projections is now presented. Again, it is a consequence of Proposition 4.13.

Corollary 4.15 *Let Δ be a differential equation with a symmetry group G possessing a normal subgroup H. Then, provided the transversality assumption is satisfied, there exists a parametrization of solutions to the* $(\dim H)$*-extended problem associated with Δ such that the HC-projected equation $\Pi_H(\Delta)$ admits a symmetry group K, with Lie algebra isomorphic to $\mathfrak{g}/\mathfrak{h}$, such that*

$$\Pi_K\left(\Pi_H(\Delta)\right) = \Pi_G(\Delta).$$

PROOF: Using Theorem 4.4 and the language of Proposition 4.13, Δ is a nondegenerate Wahlquist-Estabrook prolongation of $\Pi_G(\Delta)$ with full internal symmetry group G. $\Pi_H(\Delta)$ arises as the intermediate prolongation of $\Pi_G(\Delta)$ which was constructed in the proof of Proposition 4.13 and which is clearly an H-induced projection of the prolonged system Δ. The infinitesimal generators of K are the vector fields $\{\mathbf{v}_a^* : a = s+1, \ldots, r\}$ constructed there. □

Corollary 4.15 can be used to decompose the third order HC-projection introduced in Example 4.7 into a sequence of first order HC-projections relating various evolution equations. The details are presented below.

Example 4.16 The symmetry algebra $\mathfrak{g} = \mathrm{sp}\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$ of the heat equation which was used in Example 4.7 contains an ideal $\mathfrak{h} = \mathrm{sp}\{\mathbf{w}_1, \mathbf{w}_2\}$. Consequently, G admits a normal subgroup H and an H-induced HC-projected equation associated with the heat equation is readily shown to be

$$y_t = y^2 y_{zz} + 2y^2. \tag{4.24}$$

It is related to the heat equation by the projection

$$y = v^{-1}v_t - v^{-2}v_x^2, \quad z = -v^{-1}v_x.$$

By Corollary 4.15, the third order HC-projection from the heat equation onto equation (4.11) decomposes into the above second order projection onto equation (4.24), followed by a first order projection from that equation onto equation (4.11). The latter projection is described by $r = -yy_z$.

The HC-projection can be decomposed further, since $\mathfrak{h}$ admits an ideal $\mathfrak{i} = \mathrm{sp}\{\mathbf{w}_1\}$ leading to an I-induced HC-projected equation

$$u_t = u^2 u_{vv} \tag{4.25}$$

associated with the heat equation. This equation has already been constructed, as projection (e) of Example 3.8, and the first order projection from the heat equation is described by $u = v_x$ while the first order projection relating equations (4.24) and (4.25) is

$$y = v^{-1}uu_v - v^{-2}u^2, \quad z = -v^{-1}u.$$

The total decomposition is

$$\begin{array}{c} v_t = v_{xx} \\ \Big\downarrow{\scriptstyle I} \\ u_t = u^2 u_{vv} \\ \Big\downarrow{\scriptstyle H/I} \\ y_t = y^2 y_{zz} + 2y^2 \\ \Big\downarrow{\scriptstyle G/H} \\ r_t = r^2 r_{yy} - 2y^2 r_y + 6yr \end{array}$$

where the label to the right of each arrow indicates the symmetry group of the upper equation which induces the HC-projection onto the lower equation. These four equations, together with Burgers' equation already obtained, comprise the entire set of equations constructed by Sokolov *et al.* [87] and believed by those authors to constitute all second order, integrable, scalar evolution equations. Analysis of the method of Sokolov *et al.* appears in Appendix A. □

This section has demonstrated the ability to decompose HC-projections into sequences of lower order HC-projections when the symmetry group inducing the original projection admits a normal subgroup. Section 4.4 takes this one step further by introducing a more general type of transformation arising as one of the components of a similar decomposition of HC-projections when the symmetry group admits a subgroup which is not a normal subgroup.

4.4 M-projections

The Korteweg-de Vries (KdV) equation

$$0 = u_t + u_{xxx} + 12uu_x$$

was introduced in 1895 as a model for the evolution of long water waves down a canal of rectangular cross section [55]. Miura [65] introduced the modified KdV (mKdV) equation

$$0 = v_t + v_{xxx} - 6v^2 v_x$$

and demonstrated that it was related to the KdV equation by the simple transformation which now bears his name. Given a solution $v(x,t)$ of the mKdV equation the function $u(x,t) = \frac{1}{2}(v_x - v^2)$ satisfies the KdV equation, allowing one to project solutions of the former equation onto solutions of the latter one. Conversely, each solution $u(x,t)$ of the KdV equation can be lifted to a family of solutions to the mKdV equation by solving the system of first order equations

$$v_x = v^2 + 2u, \quad v_t = -4uv^2 - 4u_x v - 2(u_{xx} + 4u^2), \tag{4.26}$$

for $v(x,t)$. The mapping $v \mapsto u$ is known as the Miura transformation. There are several similarities between the Miura transformation and the HC-projections developed in Section 3.3, especially in the way that solutions can be mapped between the KdV and mKdV equations. Certainly, whenever $u(x,t)$ satisfies the KdV equation the system of equations (4.26) describes a solution of the one-extended problem associated with the mKdV equation. However, the set of all such solutions to the one-extended problem is not invariant under the prolongation of a nontrivial symmetry group of the mKdV equation, so that the KdV equation is not an HC-projection of the mKdV equation. Alternatively, recall that the Wahlquist-Estabrook prolongation of the KdV equation described by equations (4.26) has no nonzero internal symmetry generators and therefore does not admit a full internal symmetry group. By Theorem 4.4, the KdV equation is not an HC-projection of the prolongation defined by equations (4.26).

The Miura transformation was of prime importance in the development of "soliton" theory as it allowed Miura, Gardner and Kruskal [66] to construct the infinite family of all polynomial conservation laws of the KdV equation and, later, enabled Gardner, Greene, Kruskal and Miura [34] to motivate their discovery of the method now known as inverse scattering. It is also closely related to the bi-Hamiltonian formulation of integrable equations [58] and, as will be evident from Section 5.5, is intimately associated with auto-Bäcklund transformations of integrable equations.

With it playing such a pivotal role in soliton theory, there have been many attempts to generalize the Miura transformation and apply it to other differential equations. Sakovich [80], for instance, has determined all scalar evolution equations $\Delta^1[u]$ and $\Delta^2[v]$ related by the transformation $u(x,t) = \frac{1}{2}(v_x - v^2)$. However, there are many other transformations relating differential equations sharing most, if not all, of the properties of the Miura transformation, but having a different form. Sakovich [81] therefore generalized his transformation to $u(x,t) = \frac{1}{2}v_x + f(v)$, where f is an n-th order polynomial. Other than the equations he had obtained earlier, only trivial equations are related by this transformation. Analogues of the Miura transformation which appear in soliton theory are often much more complicated than the *Ansatz* considered by Sakovich — the transformation

$$u = v_x + \left(\frac{\tau_1}{3\alpha}\right)^{1/2} e^{v\sqrt{2\alpha}} + \left(\frac{\tau_2}{3\alpha}\right)^{1/2} e^{-v\sqrt{2\alpha}}$$

is one such example. This was found by Fokas [27] and relates a slightly generalized mKdV equation and another evolution equation involving the parameters α, τ_1 and τ_2. Thus, any attempt to generalize the Miura transformation must involve more than just testing various *Ansätze* if it is to yield the full variety of such transformations. Chapter 5 is devoted to constructing analogues of the Miura transformation. The properties of the transformations obtained there are studied in the current section.

Example 4.17 It will be shown that the KdV and mKdV equations arise as HC-projections of the PPmKdV equation

$$0 = y_t + y_{xxx} - \tfrac{3}{2} y_x^{-1} y_{xx}^2$$

corresponding to three- and two-dimensional symmetry groups, respectively, of that equation. This differential equation is usually known as the *singularity manifold equation* [93] of the KdV equation but is called the PPmKdV equation here for reasons which will explained later.

The symmetry algebra of the PPmKdV equation is six-dimensional and generated by the vector fields

$$\begin{aligned} \mathbf{v}_1 &= \partial_x, \\ \mathbf{v}_2 &= \partial_t, \\ \mathbf{v}_3 &= x\partial_x + 3t\partial_t, \\ \mathbf{v}_4 &= y\partial_y, \\ \mathbf{v}_5 &= \partial_y, \\ \mathbf{v}_6 &= y^2\partial_y. \end{aligned}$$

Let G denote the symmetry group with infinitesimal generators $\{\mathbf{v}_4, \mathbf{v}_5, \mathbf{v}_6\}$ and H the subgroup with generators $\{\mathbf{v}_4, \mathbf{v}_5\}$. Notice that the Lie algebra of G is isomorphic to $\mathfrak{sl}(2, \mathbb{R})$.

Describe solutions of the two-extended problem associated with the PPmKdV equation using local coordinates (x, t, y, y_x). Upon assuming invariance under the appropriate prolongation of H one finds that these solutions can be described by

$$y_{xx} = 2vy_x, \quad y_t = 2(v^2 - v_x)y_x,$$

where $v = v(x, t)$ must satisfy the mKdV equation. Consequently, the mKdV equation is an H-induced HC-projection of the PPmKdV equation.

Likewise, one can parametrize solutions of the three-extended problem associated with the PPmKdV equation using (x, t, y, y_x, y_{xx}) and then force invariance under the prolongation of G. Solutions are described by

$$y_{xxx} = \tfrac{3}{2} y_x^{-1} y_{xx}^2 + 4uy_x, \quad y_t = -4uy_x,$$

provided that $u = u(x, t)$ is a solution of the KdV equation. Thus, the KdV equation is a G-induced HC-projection of the PPmKdV equation. □

Using differential Galois theory, Wilson [98] has also been able to introduce $SL(2,\mathbb{R})$ and a two-dimensional subgroup into the study of the Miura transformation. In his language, the PPmKdV equation is a Galois extension of both the KdV and mKdV equations, with three- and two-dimensional Galois group respectively, but the extension of the KdV equation to the mKdV equation is not a Galois extension. This occurs because $SL(2,\mathbb{R})$ is simple, so that his two-dimensional Galois group cannot be a normal subgroup of the three-dimensional one. A similar interpretation applies to Example 4.17. Using the notation of that example, if H had been a normal subgroup of G then, from Corollary 4.15, one could immediately say that the KdV equation is a $K-$induced HC-projection of the mKdV equation for some one-dimensional symmetry group K.

The reason why the mapping of solutions between the KdV and mKdV equations mimics the mapping of solutions between equations related by an HC-projection will be explained presently. Briefly, this phenomenon occurs because the KdV and mKdV equations are HC-projected equations of a common differential equation. The essential property is that the symmetry group inducing one of these HC-projections is a subgroup of that leading to the second HC-projection. Definition 4.18 therefore describes an appropriate generalization of the Miura transformation.

Definition 4.18 Given differential equations Δ_1 and Δ_2 one says that Δ_2 is a *G/H-induced M-projected equation associated with* Δ_1 if there exists a differential equation Δ with symmetry groups G and H such that

1. $\mathfrak{h}$ is a subalgebra of $\mathfrak{g}$,
2. Δ_1 is an H-induced HC-projected equation associated with Δ and
3. Δ_2 is a G-induced HC-projected equation associated with Δ.

This M-projection is said to have *order* $\dim G - \dim H$. □

The concept of M-projections generalizes that of HC-projections since every example of the latter can be interpreted as an M-projection as defined above. Suppose that Δ_1 is a differential equation admitting a symmetry group G with Δ_2 a corresponding HC-projected equation. Introducing the differential equation $\Delta = \Delta_1$, which certainly admits symmetry groups G and $H = \{e\}$, it follows that Δ_1 and

Δ_2 are H- and G-induced HC-projected equations associated with Δ respectively. Consequently, Δ_2 is a G/H-induced M-projection of Δ_1. Returning to the general situation, whenever H is normal in G any G/H-induced M-projection of Δ_1 coincides with a K-induced HC-projection of Δ_1, where K is some symmetry group of Δ_1 with Lie algebra isomorphic to $\mathfrak{g}/\mathfrak{h}$. Thus the generalization arises from the situation with H not normal in G.

Armed with Corollary 4.15 and Definition 4.18, one can say that whenever G admits a subgroup H the G-induced HC-projection decomposes into the H-induced HC-projection, followed by either the G/H-induced HC-projection, if H is normal in G, or the G/H-induced M-projection, otherwise:

$$\begin{tikzcd} \Delta \arrow[dd, "G"'] \arrow[dr, "H"] & \\ & \Delta_1 \arrow[dl, "G/H"] \\ \Delta_2 & \end{tikzcd}$$

This observation is used in Section 4.5 to analyze the relationships between various integrable differential equations.

Each solution to Δ_2 yields a solution to the G-induced HC-projected problem associated with Δ. Proposition 4.12 shows that, subject to some technical considerations, this solution can be foliated into a $(\dim G - \dim H)$-parameter family of solutions to the H-induced HC-projected problem associated with Δ. Thus, each solution of Δ_2 can generally be lifted to a $(\dim G - \dim H)$-parameter family of solutions to Δ_1.

Each solution to Δ_1 provides a solution to the H-induced HC-projected problem associated with Δ. This will be a $(p + \dim H)$-dimensional submanifold of the appropriate jet space, where p is the number of independent variables involved in the equations Δ, Δ_1 and Δ_2. Suppose that the collection of $\mathrm{pr}^{(n)}G$-orbits through this submanifold has dimension $p + \dim G$. Then Proposition 3.7 and the comments following it show that this submanifold, or, if necessary, its first prolongation, is a solution of the G-induced HC-projected problem associated with Δ. Thus, solutions of Δ_1 satisfying certain technical conditions project onto solutions of Δ_2.

Recall from Section 4.1 that HC-projections can be interpreted as special types of Wahlquist-Estabrook prolongation. These prolongations are distinguished by the fact that they admit a full internal symmetry group as shown by Theorem 4.4. Equations related by M-projections can also be characterized in terms of special Wahlquist-Estabrook prolongations.

Continuing with the notation of Definition 4.18, provided the transversality assumption is satisfied one can interpret Δ as a nondegenerate Wahlquist-Estabrook prolongation of Δ_2 with full internal symmetry group G and involving $\dim G$ pseudopotentials. One can then construct a maximal, functionally independent set $\{x, u, y^a : a = 1, \ldots, \dim G - \dim H\}$ of invariants of the H-action. Just as in the proof of Proposition 4.13, these pseudopotentials describe a Wahlquist-Estabrook prolongation (Δ_2, Υ) of Δ_2. The prolongation equations are

$$\Upsilon_i^a = y_i^a - F_i^a(x, u^{(n-1)}, y) = 0, \quad i = 1, \ldots, p, \quad a = 1, \ldots, \dim G - \dim H, \tag{4.27}$$

where solutions to Δ_2 are described by $u(x)$. As before, one can use the infinitesimal generators of H to show that this prolongation of Δ_2 is equivalent to an H-induced HC-projection of the overall prolongation. That is, $(\Delta_2, \Upsilon) \cong \Pi_H(\Delta) = \Delta_1$. One also finds that Δ arises as a Wahlquist-Estabrook prolongation $(\Delta_2, \Upsilon, \Lambda)$ of (Δ_2, Υ) with full internal symmetry group H. All of these results are remarkably similar to those of Proposition 4.13, the only difference being that the prolongation of Δ_2 to (Δ_2, Υ) need not admit a full internal symmetry group. Thus if Δ_2 is an M-projection of Δ_1, there exists a Wahlquist-Estabrook prolongation of Δ_2 which is equivalent to Δ_1. When this prolongation does not admit a full internal symmetry group, there exists another prolongation of Δ_2 which is equivalent to the differential equation Δ introduced in Definition 4.18 and which possesses HC-projections onto both Δ_1 and Δ_2. This situation is summarized below:

$$\begin{tikzcd}
(\Delta_2, \Upsilon, \Lambda) \cong \Delta \arrow[dd, "G"'] \arrow[dr, "H"] & \\
& (\Delta_2, \Upsilon) \cong \Delta_1 \arrow[dl] \\
\Delta_2 &
\end{tikzcd}$$

Just as the Wahlquist-Estabrook prolongations associated with HC-projections ease the problem of lifting solutions from HC-projected equations to their parent differential equations, so the prolongations introduced above simplify the analogous task for M-projections. Unlike the construction for lifting solutions suggested earlier, the intermediate equation Δ need not be involved at all. All that one need do is construct, and then solve, the prolongation equations (4.27) introduced above, given that $u(x)$ is the solution to Δ_2 to be lifted. After any necessary change of coordinates, the solutions $y(x)$ to equations (4.27) will yield the required family of solutions to Δ_1. Likewise, the projection of solutions from Δ_1 to Δ_2 is most easily performed using the prolongation equations and ignoring Δ altogether.

This section concludes by generalizing another structure associated with HC-projections to one appropriate for M-projections. Section 4.2 showed that, subject to some technical considerations, a flat connection on a principal fibre bundle could be constructed for every solution to an HC-projected problem. This result can be generalized to solutions of M-projected equations, although this time the underlying fibre bundle need not be principal.

Let Δ be a system of n-th order differential equations on $M \subseteq X \times U$ with Wahlquist-Estabrook prolongations (Δ, Ξ) to $M \times Y$ and (Δ, Ξ, Υ) to $M \times Y \times Z$. Suppose further that the latter system has full internal symmetry group G with closed subgroup H such that

$$\Pi_G(\Delta, \Xi, \Upsilon) = \Delta, \quad \Pi_H(\Delta, \Xi, \Upsilon) = (\Delta, \Xi).$$

Such a situation occurs whenever Δ is an M-projection of an equation equivalent to (Δ, Ξ). Given a solution $\Phi : N \to M^{(n)}$ to Δ, the fibre bundle which often appears in work in this field is $\Phi(N) \times Y \to \Phi(N)$ and on this bundle a connection is defined by the foliation of $\Phi(N) \times Y$ into solutions of (Δ, Ξ) [86]. It will be shown that these structures are naturally inherited from the flat connection on the principal fibre bundle appropriate to HC-projections which was described in Section 4.2.

Given the above solution to Δ, the corresponding solution to the G-induced HC-projected problem associated with (Δ, Ξ, Υ) is easily shown to be $\Phi(N) \times Y \times Z$ and, since $(\Phi(N) \times Y \times Z)/G \cong \Phi(N)$ from the full internal symmetry group property, the implied principal fibre bundle is $\Phi(N) \times Y \times Z \to \Phi(N)$. Using the fact that this principal fibre bundle must be trivial, since it admits a flat connection, one can regard it simply as $\Phi(N) \times G \to \Phi(N)$. The action of G on G/H leads to an associated fibre bundle, isomorphic to the fibre bundle with total space $(\Phi(N) \times G)/H \cong \Phi(N) \times G/H$ and trivial projection $\Phi(N) \times G/H \to \Phi(N)$. This bundle inherits a connection from that on the principal G-bundle with which it is associated. It is also described by a foliation of the fibres — in this case, each leaf corresponds to the collection of H-orbits through a particular leaf of the foliation of G. Finally, recall from the proof of Proposition 4.13 that $Y \cong G/H$ and notice that the foliation of G/H just described and the foliation of Y determined above by the equations $\Xi[u, y] = 0$ coincide via this diffeomorphism.

To summarize, Section 4.2 has shown that each solution of a G-induced HC-projected equation yields a principal G-bundle on which a flat connection is described. The discussion above proves that each solution of a G/H-induced M-projected equation yields a fibre bundle, with standard fibre G/H, on which a connection can be described. This fibre bundle is associated with a principal G-bundle and the connection is inherited from a flat connection on that principal bundle.

4.5 HC-projections as an interpretive tool

This section aims to demonstrate, by way of an extended example, the application originally intended for HC-projections. A group theoretical interpretation will be given for the interrelationships between several differential equations, many of which are widely accepted as being integrable systems. In the process, several new differential equations appear which surely must also be integrable. The main goal of the example is to show that many results, which must have seemed surprising when they first appeared in the literature, have a straightforward algebraic explanation.

The starting point is the PPmKdV equation

$$0 = u_t + u_{xxx} - \tfrac{3}{2} u_x^{-1} u_{xx}^2 \tag{4.28}$$

introduced in Example 4.17, which is a key component of the M-projection relating the KdV and mKdV equations. Up to coordinate changes, all evolution equations with t as temporal variable which are related to equation (4.28) by an HC-projection will be constructed. The results of this section should be compared with those of Kalnins and Miller [48], who introduced a method determining when two evolution equations are related by an invertible transformation, as opposed to the special noninvertible ones used here.

From Example 4.17, equation (4.28) has a six-dimensional symmetry group with generators

$$\begin{aligned}\mathbf{v}_1 &= \partial_x,\\ \mathbf{v}_2 &= \partial_t,\\ \mathbf{v}_3 &= x\partial_x + 3t\partial_t,\\ \mathbf{v}_4 &= u\partial_u,\\ \mathbf{v}_5 &= \partial_u,\\ \mathbf{v}_6 &= u^2\partial_u.\end{aligned}$$

A subgroup yields an HC-projected equation which is an evolution equation in t if and only if that subgroup leaves t invariant. Thus it is necessary to classify the subalgebras of $\mathfrak{g} = \mathrm{sp}\{\mathbf{v}_1, \mathbf{v}_4, \mathbf{v}_5, \mathbf{v}_6\}$ under the adjoint representation of the full symmetry group on $\mathfrak{g}$.

Consider the one-dimensional subalgebra of $\mathfrak{g}$ spanned by the vector

$$\mathbf{u} = a\mathbf{v}_1 + b\mathbf{v}_4 + c\mathbf{v}_5 + d\mathbf{v}_6, \quad a, b, c, d \in \mathbb{R}.$$

When $a = 0$ only two nonconjugate subalgebras are found, denoted by $\mathfrak{j}_1 = \mathrm{sp}\{\mathbf{v}_4\}$ and $\mathfrak{j}_2 = \mathrm{sp}\{\mathbf{v}_5\}$ respectively. This result merely reflects the fact that $\mathfrak{sl}(2, \mathbb{R}) \cong \mathrm{sp}\{\mathbf{v}_4, \mathbf{v}_5, \mathbf{v}_6\}$ has two nonconjugate subalgebras of dimension one. Furthermore, since $\mathfrak{g} = \mathrm{sp}\{\mathbf{v}_1\} \oplus \mathrm{sp}\{\mathbf{v}_4, \mathbf{v}_5, \mathbf{v}_6\}$ is a Lie algebra direct sum, it follows that there exist exactly three other one-dimensional subalgebras up to conjugation. They are $\mathfrak{j}_3 = \mathrm{sp}\{\mathbf{v}_1 + \mathbf{v}_5\}$, $\mathfrak{j}_4 = \mathrm{sp}\{\mathbf{v}_1 + \mathbf{v}_4\}$ and $\mathfrak{j}_5 = \mathrm{sp}\{\mathbf{v}_1\}$. Thus, up to changes of coordinates, there exist exactly five evolution equations related to equation (4.28) by a first order HC-projection.

For each $k = 1, \ldots, 5$ let J_k denote the symmetry group of equation (4.28) with Lie algebra $\mathfrak{j}_k$. Then a J_1-induced HC-projected equation associated with the PPmKdV equation is found to be

$$0 = v_t + v_{xxx} - \tfrac{1}{2}v_y^3 - \tfrac{3}{2}v_y e^{2v}, \tag{4.29}$$

which is a special case of a family of integrable equations studied by Calogero and Degasperis [7]. The projection is efficiently described by $v = \log(u_x/u)$ and $y = x$. Next, one finds that the PmKdV equation (this name will be motivated in Example 5.3)

$$0 = v_t + v_{yyy} - 2v_y^3$$

is a J_2-induced HC-projected equation, with the HC-projection given by $v = \frac{1}{2}\log u_x$ and $y = x$. The equation corresponding to the symmetry group J_3 is

$$0 = 2v^2 v_t + 2v^2(v-1)^3 v_{yyy} + 6v(v-1)^2 v_y v_{yy} - 3(v-1)^2 v_y^3 \tag{4.30}$$

and is related to equation (4.28) by $v = u_x$ and $y = u - x$. J_4 yields the HC-projected equation

$$0 = 2v^2 v_t + 2v^2(v-1)^3 v_{yyy} + 6v(v-1)^2 v_y v_{yy} - 3(v-1)^2 v_y^3 + (3-2v)v^4 v_y \tag{4.31}$$

with projection described by $v = u^{-1}u_x$ and $y = \log u - x$. Finally, the Harry Dym equation

$$0 = v_t + v^3 v_{yyy}$$

occurs as a J_5-induced HC-projected equation associated with equation (4.28). The projection can be described simply by $v = u_x$ and $y = u$.

Construction of all second order HC-projections of the PPmKdV equation is eased by recalling that every two-dimensional Lie algebra is solvable. Thus, each two-dimensional subalgebra of $\mathfrak{g}$ must be conjugate to one containing an ideal equal to one of $\mathfrak{j}_1, \dots, \mathfrak{j}_5$. Since the normalizer of $\mathfrak{j}_1$ in the full symmetry algebra is $\mathfrak{n}[\mathfrak{j}_1] = \mathrm{sp}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$, it follows that the most general two-dimensional subalgebra containing $\mathfrak{j}_1$ as an ideal and which generates a symmetry group leaving t invariant is $\mathrm{sp}\{\mathbf{v}_1, \mathbf{v}_4\}$. This process can be repeated with the remaining one-dimensional subalgebras obtained above, yielding a system of four two-dimensional subalgebras of $\mathfrak{g}$. They are

$$\mathfrak{i}_1 = \mathrm{sp}\{\mathbf{v}_4, \mathbf{v}_5\}, \quad \mathfrak{i}_2 = \mathrm{sp}\{\mathbf{v}_1 + \mathbf{v}_4, \mathbf{v}_5\}, \quad \mathfrak{i}_3 = \mathrm{sp}\{\mathbf{v}_1, \mathbf{v}_5\}, \quad \mathfrak{i}_4 = \mathrm{sp}\{\mathbf{v}_1, \mathbf{v}_4\},$$

and generate symmetry groups $I_1, \dots, I_4$ of the PPmKdV equation respectively.

Up to coordinate changes, there are thus four evolution equations related to equation (4.28) by a second order HC-projection. The first one is the well known mKdV equation

$$0 = w_t + w_{zzz} - 6w^2 w_z,$$

which is related by the I_1-induced HC-projection $w = \frac{1}{2}u_x^{-1}u_{xx}$ and $z = x$. I_2 yields the projected equation

$$0 = w_t + w^3 w_{zzz} + 3w^2 w_z w_{zz} + 2(1+w)^2(1-2w)w_z, \tag{4.32}$$

with the projection from the PPmKdV equation being given by $w = \frac{1}{2}u_x^{-1}u_{xx} - 1$ and $z = \frac{1}{2}\log u_x - x$. The HC-projected equation

$$0 = w_t + w^3 w_{zzz} + 3w^2 w_z w_{zz} - 4w^3 w_z \tag{4.33}$$

induced by the symmetry group I_3 has appeared before as equation (3.18) in Example 3.12. It is a vital component of the auto-Bäcklund transformation which was constructed in that example for the Harry Dym equation and is related to the PPmKdV equation by $w = \frac{1}{2}u_x^{-1}u_{xx}$ and $z = \frac{1}{2}\log u_x$. The final evolution equation related by a second order projection corresponds to I_4 and is

$$0 = w_t + w^3 w_{zzz} + 3w^2 w_z w_{zz} - w^3 w_z - 3e^{2z}w^2, \tag{4.34}$$

where $w = u_x^{-1}u_{xx} - u^{-1}u_x$ and $z = \log(u_x/u)$.

Since every three-dimensional Lie algebra is either simple or solvable, other than $\mathrm{sp}\{\mathbf{v}_4, \mathbf{v}_5, \mathbf{v}_6\} \cong \mathfrak{sl}(2,\mathbb{R})$ all three-dimensional subalgebras of $\mathfrak{g}$ must be conjugate to an algebra containing an ideal equal to one of $\mathfrak{i}_1, \ldots, \mathfrak{i}_4$. It quickly follows that every three-dimensional subalgebra of $\mathfrak{g}$ is conjugate to either $\mathfrak{h}_1 = \mathrm{sp}\{\mathbf{v}_4, \mathbf{v}_5, \mathbf{v}_6\}$ or $\mathfrak{h}_2 = \mathrm{sp}\{\mathbf{v}_1, \mathbf{v}_4, \mathbf{v}_5\}$. Taking H_1 and H_2 as the corresponding symmetry groups of equation (4.28), one finds that an H_1-induced HC-projection of the PPmKdV equation is

$$0 = p_t + p_{rrr} + 12pp_r,$$

the KdV equation, with projection $p = \frac{1}{4}u_x^{-1}u_{xxx} - \frac{3}{8}u_x^{-2}u_{xx}^2$ and $r = x$. The H_2-induced projected equation

$$0 = p_t + p^3 p_{rrr} + 3p^2 p_r p_{rr} - 12rp^2 \tag{4.35}$$

is new and related to equation (4.28) by $p = \frac{1}{2}u_x^{-1}u_{xxx} - \frac{1}{2}u_x^{-2}u_{xx}^2$ and $r = \frac{1}{2}u_x^{-1}u_{xx}$. All that remains is the G-induced projection, where G is the symmetry group corresponding to $\mathfrak{g}$. This equation also seems to be new. It is

$$0 = q_t + q^3 q_{sss} + 3q^2 q_s q_{ss} + 12q^2, \tag{4.36}$$

where $q = \frac{1}{4}u_x^{-1}u_{xxxx} - u_x^{-2}u_{xx}u_{xxx} + \frac{3}{4}u_x^{-3}u_{xx}^3$ and $s = \frac{1}{4}u_x^{-1}u_{xxx} - \frac{3}{8}u_x^{-2}u_{xx}^2$.

Twelve differential equations have been constructed above and, up to coordinate changes, they constitute all evolution equations with t as temporal variable which are HC-projections of the PPmKdV equation. One could stop this example here, content to have constructed several new differential equations all related to one well-studied equation. Although these new equations have no obvious physical application, they are surely examples of completely integrable systems, since they are so closely related to the prototypical KdV equation. Therefore, one could regard the approach of constructing HC-projections of such an equation as being akin to a "soliton factory" — yielding new equations on which to try out any conjectured tests for integrability. Similar work, but using different methods, has been performed by Fokas [27] and Ibragimov and Shabat [46], who determined evolution equations admitting generalized symmetries, and Leo, Leo, Soliani, Solombrino and Martina [62], who identified evolution equations admitting non-Abelian Wahlquist-Estabrook prolongation algebras.

However, HC-projections were devised to do more than just generate new differential equations. In particular, they were intended to help study many diverse structures associated with differential equations, especially integrable ones, by allowing a group-theoretic interpretation. Therefore, it is appropriate to continue this example by first considering the algebraic structure of the symmetry algebra $\mathfrak{g}$ on which the whole construction is based. Figure 4.1 is the subalgebra lattice diagram for this Lie algebra, with all of the subalgebras considered here being displayed. An arrow joining two such algebras indicates that the "target" algebra is a subalgebra of the "source" algebra. A solid arrow indicates that this is in fact an ideal, a point which is important when one wishes to distinguish between HC- and M-projections. Notice, though, that no attempt has been made to illustrate this relationship when the dimension of the factor algebra is greater than one. This is necessary to keep the figure legible. One can construct a similar lattice diagram to represent the HC-projections of the PPmKdV equation derived from the symmetry algebras displayed in Figure 4.1. The result is shown in Figure 4.2, where an arrow, broken or solid, pointing from one equation to another indicates that the "target" equation is an M- or HC-projection, respectively, of the "source" equation. Once more, for clarity, the higher order HC-projections constructed earlier in this section have been decomposed into sequences of first order HC- and M-projections, as approriate.

Figures 4.1 and 4.2 lead to some interesting reinterpretations of known results. In

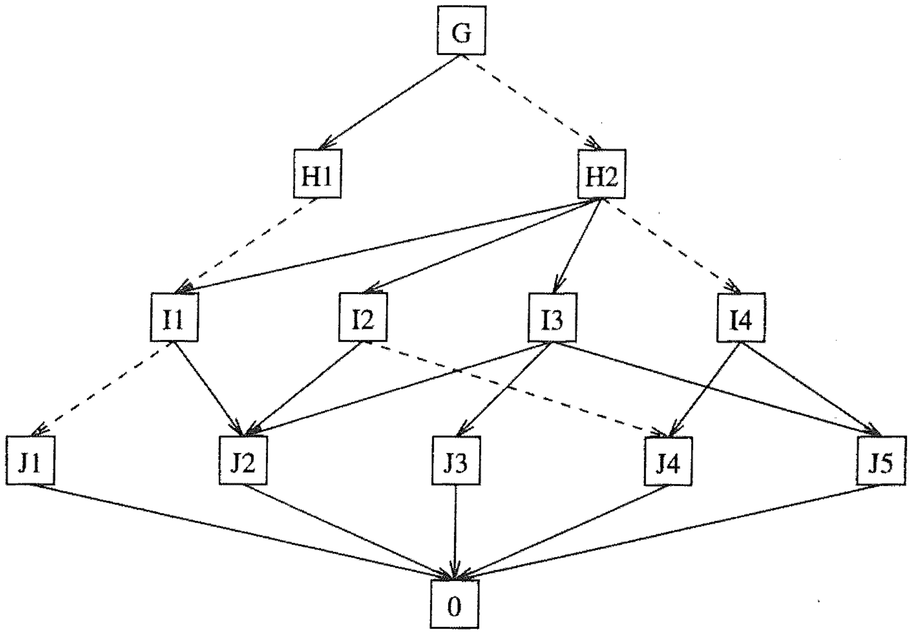

Figure 4.1: Subalgebra lattice diagram for the Lie algebra $\mathfrak{g} = \mathrm{sp}\{\mathbf{v}_1, \mathbf{v}_4, \mathbf{v}_5, \mathbf{v}_6\}$

recent years, many papers have been published which analyze the relationships between several of the equations which feature in this lattice diagram. Included among these are the pioneering paper of Miura [65] which first presented the M-projection relating the KdV and mKdV equations and a more recent study by Weiss [93] which introduced the HC-projection linking the KdV and PPmKdV equations. One sees from Figures 4.1 and 4.2 that these transformations derive from the fact that equation (4.28) admits a symmetry group isomorphic to $SL(2,\mathbb{R})$. Several papers have appeared, beginning with the one by Kawamoto [51], which feature complicated transformations relating solutions of the Harry Dym equation and the mKdV equation. Armed with Figure 4.2, the origin of these transformations is obvious — they arise from the fact that the Harry Dym equation and the mKdV equation are HC-projections of a common differential equation, a point first made by Ibragimov [45]. The procedures advocated by the above authors for mapping solutions from, say, the mKdV equation to the Harry Dym equation thus amount to obtaining the two-parameter family of solutions of the PPmKdV equation and then projecting these onto a family of solutions to the Harry Dym equation.

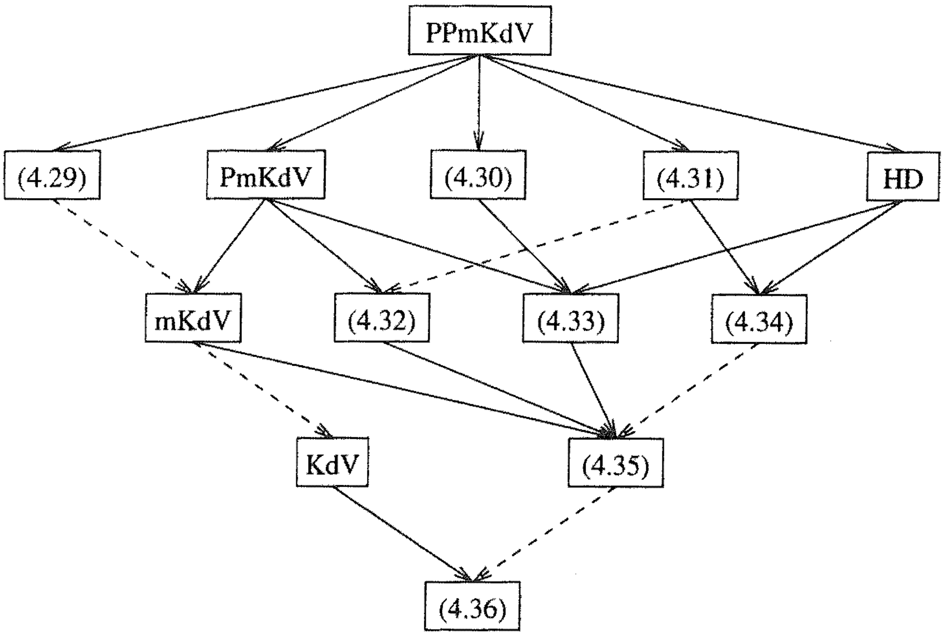

Figure 4.2: Lattice diagram representing the HC-projected evolution equations associated with the PPmKdV equation

Figure 4.2 also provides a better understanding of the auto-Bäcklund transformation derived for the Harry Dym equation in Example 3.12. Recall that the essential components of that auto-Bäcklund transformation were the HC-projection from the Harry Dym equation onto equation (4.33), together with a one-dimensional symmetry group of the former equation and a discrete symmetry group of the latter one. It turns out that all three objects can be pulled back to the system comprising the PPmKdV and PmKdV equations. The appropriate part of the lattice diagram is shown below:

$$\begin{tikzcd} & 0 = u_t + u_{xxx} - \frac{3}{2}u_x^{-1}u_{xx}^2 \arrow[dl] \arrow[dr] & \\ 0 = v_t + v_{yyy} - 2v_y^3 \arrow[dr] & & 0 = v_t + v^3 v_{yyy} \arrow[dl] \\ & 0 = w_t + w^3 w_{zzz} + 3w^2 w_z w_{zz} - 4w^3 w_z & \end{tikzcd}$$

The one-dimensional symmetry group $\exp(a\mathbf{v}_6)$ of the PPmKdV equation acts on solutions via

$$\exp(a\mathbf{v}_6) : u(x,t) \mapsto \tilde{u}(x,t) = \frac{u(x,t)}{1 - au(x,t)},$$

whenever $\tilde{u}(x,t)$ is defined, and projects onto the continuous symmetry group

$$v(y,t) \mapsto \tilde{v}(y,t) = (1+ay)^2 \cdot v\left(\frac{y}{1+ay}, t\right)$$

of the Harry Dym equation. Furthermore, a discrete symmetry group of the PmKdV equations acts on solutions of that equation, almost trivially, as

$$k : v(y,t) \mapsto \tilde{v}(y,t) = -v(y,t)$$

and projects onto the discrete symmetry group

$$w(z,t) \mapsto \tilde{w}(z,t) = -w(-z,t)$$

of equation (4.33). It follows from these observations and the commutativity of the above diagram that the auto-Bäcklund transformation derived in Example 3.12 has a counterpart for the system comprising the PPmKdV and PmKdV equations. This auto-Bäcklund transformation is still described by

$$\begin{array}{ccccccc} u_0(x,t) & \xrightarrow{\exp(a\mathbf{v}_6)} & u_1(x,t) & & u_2(x,t) & \xrightarrow{\exp(-a\mathbf{v}_6)} & u_3(x,t) \\ & & \downarrow & & \downarrow & & \\ & & v_1(y,t) & \xrightarrow{k} & v_2(y,t) & & \end{array}$$

although the equations and transformations which it represents have changed. In fact, it is much easier to study this auto-Bäcklund transformation because, as can be seen from the form of $\exp(a\mathbf{v}_6)$, k and the HC-projection, there will never be any need to describe solutions to the equations implicitly as was the case for the other system. This matter will not be pursued here, but it is mentioned in passing that, when applied to the seed solution $u = x$ of the PPmKdV equation, this auto-Bäcklund transformation appears to generate the infinite family of solutions which project onto the rational solutions of the KdV and mKdV equations. For instance, the solution of the PPmKdV equation obtained after application of the method above to the seed solution is

$$u(x,t) = \frac{(ax-1)^3 + 12a^3t + 3ab}{3a + a\left((ax-1)^3 + 12a^3t + 3ab\right)}.$$

Finally, a suggestive relationship is immediately apparent between Figures 4.1 and 4.2. One is able to obtain the lattice of HC-projected equations from the subalgebra lattice by simply inverting the latter diagram, reversing the direction of all arrows and replacing the subalgebras by the HC-projected equations which the corresponding connected symmetry groups induce. This may remind one of Galois theory and its interplay of group and field theory. To strengthen the analogy, one should reinterpret the lattice diagram of projected equations. Recall, from Section 4.1, that each HC-projection yields a nondegenerate Wahlquist-Estabrook prolongation of the projected equation, with full internal symmetry group, which recovers the original differential equation. In terms of Figure 4.2, one can say that for any two equations connected by an arrow, the "source" equation arises as a Wahlquist-Estabrook prolongation of the "target" equation. In this interpretation Wahlquist-Estabrook prolongations of differential equations are the analogues of the extension fields of Galois theory and Galois groups correspond to the internal symmetry groups of the prolongations. Finite normal extensions correspond to nondegenerate Wahlquist-Estabrook prolongations with full internal symmetry group. Many of the results well known from Galois theory then carry over to yield analogous results for these Wahlquist-Estabrook prolongations. As one example, in Galois theory an intermediate field is a normal extension of the field F if and only if a certain group is a normal subgroup of the overall Galois group. The corresponding result here says that an intermediate prolongation of equation (4.36) has full internal symmetry group if and only if the internal symmetry group of the prolongation of this prolonged system to the PPmKdV equation is a normal subgroup of G. This has been proved in one direction already, in Proposition 4.13. From the Galois analogy one would expect the converse result to also be true, although the author is currently unable to prove this result. As an example, consider the prolongation of equation (4.36) to the mKdV equation. From Figures 4.1 and 4.2, the mKdV equation is an I_1-induced HC-projection of equation (4.28), but I_1 is not a normal subgroup of G, since $\mathfrak{i}_1$ is not an ideal of $\mathfrak{g}$. Thus, the prolongation of equation (4.36) yielding the mKdV equation would not have full internal symmetry group. This can be confirmed by direct calculation. Equivalently, equation (4.36) is an M-projection, and not an HC-projection, of the mKdV equation.

Hopefully, this section has shown that HC-projections can be a valuable tool when analyzing many structures associated with differential equations. They certainly provide a useful setting in which to analyze the role of group theory as it relates to integrable systems of differential equations.

Chapter 5

Constructing M-projections

This chapter concentrates on the M-projections introduced in Section 4.4. One problem is to construct M-projections given only information concerning the analogue of the mKdV equation. Section 5.1 restates this problem in terms of Wahlquist-Estabrook prolongations and their symmetry properties. By suitably generalizing the notion of symmetry generator, Section 5.2 is able to develop a reasonably straightforward, if not systematic, technique for constructing Wahlquist-Estabrook prolongations suitable for deriving M-projections. Often, when the equation being studied admits a recursion operator this technique can be improved using the refined recursion operators discussed in Section 5.3. The next section summarizes the steps involved in constructing M-projections, as well as presenting a more efficient means of performing some of the calculations involved. Finally, Section 5.5 returns to the problem of constructing Bäcklund transformations for differential equations. Combining the results of Sections 3.3 and 5.2, the prolongation approach of Wahlquist and Estabrook to the problem is significantly enhanced.

5.1 Constructing M-projections: the problem

M-projections, as described thus far, are only useful *a posteriori* in analyzing generalizations of the Miura transformation. For instance, Example 4.17 did not derive the Miura transformation from the mKdV equation, it simply displayed how it could be associated with suitable HC-projections of the PPmKdV equation. In terms of

Definition 4.18, the starting point when constructing M-projections has been the intermediate differential equation Δ which admits both Δ_1 and Δ_2 as HC-projections, indicating that Δ_2 is an M-projection of Δ_1. However, in practice one will begin with either Δ_1 or Δ_2 instead of Δ. One then wishes to be able to construct Δ_2 or Δ_1 respectively. Given Δ_2, the discussion in Section 4.4 shows that Δ_1 arises as a Wahlquist-Estabrook prolongation of Δ_2, so that one can construct "source" equations from "target" equations using the prolongation method. This generalizes a similar result which holds for HC-projections, as discussed in Section 4.1. Therefore, the outstanding problem is to construct the target equation Δ_2 of the M-projection from knowledge only of the source equation Δ_1. The current chapter examines this problem in detail, with the problem being formally stated below.

Problem A Given a differential equation Δ one requires a differential equation $\tilde{\Delta}$ admitting symmetry groups G and H such that

1. $\mathfrak{h}$ is a subalgebra of $\mathfrak{g}$, but not an ideal, and
2. Δ is an H-induced HC-projected equation associated with $\tilde{\Delta}$. □

Having solved this problem, one immediately obtains a new differential equation, a G-induced HC-projected equation associated with $\tilde{\Delta}$, which is a G/H-induced M-projection of Δ. The restriction that $\mathfrak{h}$ not be an ideal in $\mathfrak{g}$ is included so as to eliminate the possibility of the new equation being an HC-projection of the original one. Clearly, HC-projections of Δ can be constructed without the need to identify the equation $\tilde{\Delta}$!

The fact that the equation $\tilde{\Delta}$ sought in Problem A must admit the original differential equation Δ as an HC-projection indicates that Δ must admit a nondegenerate Wahlquist-Estabrook prolongation equivalent to $\tilde{\Delta}$ with full internal symmetry group. The approach advocated here for solving Problem A involves such prolongations intimately and depends crucially on the symmetry groups admitted by these prolongations. Some terminology used to describe the symmetry generators of a Wahlquist-Estabrook prolongation is presented in the following definition.

Definition 5.1 Let $\Delta[u] = 0$ denote a system of differential equations defined on M with Wahlquist-Estabrook prolongation (Δ, Ξ) to $M \times Y$. The trivial projection $\pi_1 : M \times Y \to M$ allows one to group the infinitesimal symmetry generators of

(Δ, Ξ) into three categories. Let the vector field $\mathbf{v}$ on $M \times Y$ generate a symmetry group of the system of differential equations (Δ, Ξ).

1. Whenever $(\pi_1)_*\mathbf{v}$ exists $\mathbf{v}$ is said to be an *inherited symmetry generator* of (Δ, Ξ).

2. If $(\pi_1)_*\mathbf{v} = 0$ then $\mathbf{v}$ is called an *internal symmetry generator* of (Δ, Ξ).

3. If $\mathbf{v}$ is not π_1-projectable one says that it is a *nonlocal symmetry generator* of (Δ, Ξ). Examples of such vector fields can be found in [5] and [52]. □

Internal symmetry generators have already been defined in Definition 4.2. Inherited symmetry generators are a generalization, since they certainly contain internal ones as a special case. The name "inherited" is used since it is easily shown that if $\mathbf{v}$ is an inherited symmetry generator then the vector field $(\pi_1)_*\mathbf{v}$ on M generates a symmetry group of Δ. Another simple proof shows that the set of inherited symmetry generators forms a subalgebra of the symmetry algebra of (Δ, Ξ) and that the set of internal symmetry generators is an ideal of this subalgebra. Thus, the *inherited symmetry algebra* and *internal symmetry algebra* of a Wahlquist-Estabrook prolongation are well-defined structures, as are the corresponding inherited and internal (connected) symmetry groups. The term "nonlocal" symmetry generator will be motivated later in the current section. It is remarked, in passing, that the commutator of two nonlocal symmetry generators of (Δ, Ξ) need not remain a nonlocal symmetry generator, so it makes no sense to consider the nonlocal symmetry algebra of a Wahlquist-Estabrook prolongation.

The following is a restatement of Problem A, using the language of Wahlquist-Estabrook prolongations and Definition 5.1.

Problem B Given a differential equation Δ one requires a Wahlquist-Estabrook prolongation of Δ which is nondegenerate and admits both a full internal symmetry group and a nonlocal symmetry generator. □

The precise relationship between Problems A and B is given in the following theorem.

Theorem 5.2 *If (Δ, Ξ) is a solution to Problem B then the differential equation $\tilde{\Delta} = (\Delta, \Xi)$ is a solution of Problem A. Conversely, suppose that $\tilde{\Delta}$ is a solution to*

Problem A. Then there exists a Wahlquist-Estabrook prolongation (Δ, Ξ) equivalent to $\tilde{\Delta}$ such that (Δ, Ξ) is a solution of Problem B.

PROOF: Suppose that (Δ, Ξ) is a solution to Problem B, where Δ is a system of differential equations on $M \subseteq X \times U$ and (Δ, Ξ) is a Wahlquist-Estabrook prolongation to $M \times Y$. There are two obvious symmetry groups of $\tilde{\Delta} = (\Delta, \Xi)$. Let H denote the internal symmetry group of (Δ, Ξ) and let G be a local group of transformations with Lie algebra generated by the internal symmetry generators and some nonempty set of nonlocal symmetry generators. Then $\mathfrak{h}$ is a subalgebra of $\mathfrak{g}$ and, by Theorem 4.4, Δ is an H-induced HC-projected equation associated with $\tilde{\Delta}$.

It remains to prove that $\mathfrak{h}$ is not an ideal in $\mathfrak{g}$. Taking $x = (x^1, \ldots, x^p)$, $u = (u^1, \ldots, u^q)$ and $y = (y^1, \ldots, y^r)$ as coordinates for $X = \mathbb{R}^p$, $U = \mathbb{R}^q$ and $Y = \mathbb{R}^r$ respectively, let

$$\mathbf{u} = \sum_{i=1}^{p} \xi^i(x,u,y)\partial_{x^i} + \sum_{\alpha=1}^{q} \phi^\alpha(x,u,y)\partial_{u^\alpha} + \sum_{a=1}^{r} \psi^a(x,u,y)\partial_{y^a} \tag{5.1}$$

be one of the nonlocal symmetry generators in $\mathfrak{g}$ and suppose that $\mathfrak{h}$ is an ideal in $\mathfrak{g}$. Then $[\mathbf{u}, \mathbf{v}]$ must be an internal symmetry generator for all internal symmetry generators $\mathbf{v}$. The most general such vector field is

$$\mathbf{v} = \sum_{a=1}^{r} \eta^a(x,u,y)\partial_{y^a}, \tag{5.2}$$

so that

$$[\mathbf{u}, \mathbf{v}] = -\sum_{i=1}^{p} \mathbf{v}(\xi^i)\partial_{x^i} - \sum_{\alpha=1}^{q} \mathbf{v}(\phi^\alpha)\partial_{u^\alpha} + \sum_{a=1}^{r} (\mathbf{u}(\eta^a) - \mathbf{v}(\psi^a))\partial_{y^a}. \tag{5.3}$$

If $[\mathbf{u}, \mathbf{v}]$ is to be an internal symmetry generator, then

$$\mathbf{v}(\xi^i) = \mathbf{v}(\phi^\alpha) = 0, \quad i = 1, \ldots, p, \ \ \alpha = 1, \ldots, q.$$

Since this must hold for all internal symmetry generators $\mathbf{v}$ it follows from the definition of a full internal symmetry group that $\xi^i = \xi^i(x,u)$ and $\phi^\alpha = \phi^\alpha(x,u)$, so that $\mathbf{u}$ is π_1-projectable. Therefore, $\mathfrak{h}$ cannot be an ideal in $\mathfrak{g}$.

Conversely, suppose that $\tilde{\Delta}$ is a solution to Problem A and, by suitably restricting domains until the transversality requirement is satisfied, let (Δ, Ξ) denote the Wahlquist-Estabrook prolongation of $\Delta = \Pi_H(\tilde{\Delta})$ featured in Theorem 4.4. It is sufficient to prove that this prolongation admits a nonlocal symmetry generator. Since

$\mathfrak{h}$ cannot be an ideal in $\mathfrak{g}$, there exist vectors $\mathbf{u} \in \mathfrak{g}$ and $\mathbf{v} \in \mathfrak{h}$ such that $[\mathbf{u}, \mathbf{v}] \notin \mathfrak{h}$. That is, $[\mathbf{u}, \mathbf{v}]$ is not an internal symmetry generator of (Δ, Ξ). Using the same coordinates as in the first part of this proof, and the fact that $\mathbf{v}$ is an internal symmetry generator, one can let $\mathbf{u}$ and $\mathbf{v}$ be described by equations (5.1) and (5.2) respectively. Since their Lie bracket, given by equation (5.3), cannot be an internal symmetry generator, at least one of the functions $\{\mathbf{v}(\xi^i), \mathbf{v}(\phi^\alpha) : i = 1, \ldots, p,\ \alpha = 1, \ldots, q\}$ must be nonzero. But if $\mathbf{u}$ is an inherited symmetry generator then all of these functions will vanish, proving that $\mathbf{u}$ is nonlocal. □

Sections 5.2 and 5.3 develop techniques for solving Problem B. It will prove very useful to consider the following example, which introduces a Wahlquist-Estabrook prolongation of the mKdV equation meeting the requirements of Problem B. This prolongation is then used to construct the Miura transformation, recovering the KdV equation in the process.

Example 5.3 The mKdV equation

$$0 = v_t + v_{xxx} - 6v^2 v_x,$$

which will be referred to as Δ for the remainder of this example, is described on $M = X \times U$, where $X = \mathbb{R}^2$ and $U = \mathbb{R}^1$ have coordinates (x, t) and v respectively. Δ admits a potential w which is defined by the equations

$$w_x = v, \quad w_t = -v_{xx} + 2v^3. \tag{5.4}$$

The system of equations comprising Δ together with equations (5.4) possesses a further potential y defined by the equations

$$y_x = e^{2w}, \tag{5.5}$$

$$y_t = 2e^{2w}(-v_x + v^2). \tag{5.6}$$

Introducing $Y = \mathbb{R}^2$ with coordinates (w, y) it is easily shown that (Δ, Ξ) is a Wahlquist-Estabrook prolongation of Δ where Ξ represents equations (5.4) to (5.6).

It is now possible to explain why the equation

$$0 = y_t + y_{xxx} - \tfrac{3}{2} y_x^{-1} y_{xx}^2, \tag{5.7}$$

which appeared in Example 4.26, is being called the PPmKdV equation. A straightforward calculation confirms that the function $w(x,t)$ featured in the prolongation (Δ, Ξ) must be a solution of

$$0 = w_t + w_{xxx} - 2w_x^3.$$

Since w is a potential of the mKdV equation this equation is usually called the potential mKdV equation and abbreviated to the PmKdV equation. Now y can be interpreted as a potential of the PmKdV equation defined by equation (5.5) and

$$y_t = 2e^{2w}(-w_{xx} + w_x^2),$$

where the latter equation has been obtained from equation (5.6) via the substitution $v = w_x$. Another simple calculation shows that $y(x,t)$ must be a solution to equation (5.7). Consequently, equation (5.7) will be called the potential PmKdV equation, abbreviated to the PPmKdV equation.

The symmetry algebra of (Δ, Ξ) is six-dimensional and generated by the vector fields

$$\begin{aligned}
\mathbf{v}_1 &= \partial_x, \\
\mathbf{v}_2 &= \partial_t, \\
\mathbf{v}_3 &= x\partial_x + 3t\partial_t - v\partial_v + y\partial_y, \\
\mathbf{v}_4 &= \partial_w + 2y\partial_y, \\
\mathbf{v}_5 &= \partial_y, \\
\mathbf{v}_6 &= e^{2w}\partial_v + y\partial_w + y^2\partial_y.
\end{aligned}$$

One sees that $\{\mathbf{v}_1, \ldots, \mathbf{v}_5\}$ is a basis for the inherited symmetry algebra of (Δ, Ξ) and that the internal symmetry group has infinitesimal generators $\mathbf{v}_4$ and $\mathbf{v}_5$. The vector field $\mathbf{v}_6$ is an example of a nonlocal symmetry generator.

A smooth function θ on $M^{(n-1)} \times Y$ satisfies $\tilde{D}_x\theta = 0$ on solutions to the mKdV equation if and only if $\theta = \theta(t)$. Thus (Δ, Ξ) is nondegenerate. For an arbitrary smooth function f on $M \times Y$, $\mathbf{v}_5(f) = 0$ implies that f is independent of y and $\mathbf{v}_4(f) = 0$ then implies that it is independent of w. That is, f is independent of the pseudopotentials and (Δ, Ξ) admits a full internal symmetry group. It follows from Theorem 4.4 that the mKdV equation arises as an HC-projected equation associated with (Δ, Ξ) and induced by the internal symmetry group.

A more interesting HC-projection results from using the symmetry group G with infinitesimal generators $\{\mathbf{v}_4, \mathbf{v}_5, \mathbf{v}_6\}$. Taking (x, t, v, w, y) as independent variables for the three-extended equation associated with (Δ, Ξ), one can simplify the resulting system of equations to a single, third order differential equation for $v_x = v_x(x, t, v, w, y)$. Solutions of this equation which are invariant under the extension of G take the special form

$$v_x(x, t, v, w, y) = v^2 + 2u(x, t), \tag{5.8}$$

where $u(x, t)$ must satisfy the KdV equation

$$0 = u_t + u_{xxx} + 12uu_x.$$

Thus, beginning with the mKdV equation one is able to recover the KdV equation. The Miura transformation relating these two equations is obtained by solving equation (5.8) for $u(x, t)$. □

Because $\mathbf{v}_6$ is not π_1-projectable, the group it generates will not project onto a symmetry group of the mKdV equation in the classical sense. Of course, that group will actually be a symmetry group of the prolongation (Δ, Ξ) of the mKdV equation and so takes solutions of the prolonged system into other solutions of (Δ, Ξ). This fact can be used to generate new solutions of the mKdV equation as follows. Using an initial solution $v(x, t)$ of the mKdV equation, solve equations (5.4) to (5.6) and apply the symmetry group generated by $\mathbf{v}_6$ to this solution of (Δ, Ξ). This will yield new solutions for the three dependent variables involved in the prolonged system. If one is only interested in solutions to the mKdV equation, the new values for the pseudopotentials may be ignored. One can easily prove that the new solution of the mKdV equation is given by

$$\exp(a\mathbf{v}_6) : v(x, t) \mapsto \tilde{v}(x, t) = v(x, t) + \frac{a \cdot e^{2w(x,t)}}{1 - a \cdot y(x, t)}$$

whenever $\tilde{v}(x, t)$ is defined. Since this new solution involves the pseudopotentials, or "nonlocal" variables, w and y, the mapping $v(x, t) \mapsto \tilde{v}(x, t)$ is called an application of a nonlocal symmetry group of the mKdV equation and $\mathbf{v}_6$ is known as a nonlocal symmetry generator.

One way to think of the construction of M-projections is as a generalization of the method of HC-projections, with the (classical) symmetry groups of the earlier method being replaced by nonlocal symmetry groups.

5.2 Partial symmetry generators

As explained in the preceding section, construction of M-projected equations associated with a differential equation Δ requires the existence of a Wahlquist-Estabrook prolongation of Δ admitting a nonlocal symmetry generator. Unfortunately, not all prolongations share this property.

Example 5.4 The system of equations

$$w_x = v, \quad w_t = -v_{xx} + 2v^3, \tag{5.9}$$

defines a Wahlquist-Estabrook prolongation of the mKdV equation

$$0 = v_t + v_{xxx} - 6v^2 v_x.$$

Calculations confirm that this prolongation does not admit any nonlocal symmetry generators. □

One would like a method capable of predicting exactly which prolongations of a differential equation admit nonlocal symmetry generators. At the moment, all one can do is calculate the symmetry algebra for each prolongation of the equation in turn, until a prolongation is discovered with the required nonlocal symmetry generator. The approach of the current section is to generalize the notion of nonlocal symmetry generators slightly, so that a larger class of Wahlquist-Estabrook prolongations is of interest. Of course, not all of these prolongations will admit a genuine nonlocal symmetry generator. Later in this section, a technique will be developed which is in principle capable of augmenting such prolongations to larger ones which do admit a true nonlocal symmetry generator. This may not seem to have improved the situation a great deal, because one is still forced to test various prolongations of the given differential equation. But by casting the net wider, in this case by looking for prolongations admitting the more general type of symmetry, the chances of success are correspondingly higher. When the differential equation being studied possesses a recursion operator, it is sometimes possible to construct Wahlquist-Estabrook prolongations for which the existence of these new symmetries is guaranteed. This situation is examined in Section 5.3.

Let $\Delta[u] = 0$ denote a system of n-th order differential equations on $M \subseteq X \times U$ and suppose that (Δ, Ξ) is a Wahlquist-Estabrook prolongation to $M \times Y$. Suppose that $X = \mathbb{R}^p$, $U = \mathbb{R}^q$ and $Y = \mathbb{R}^r$ have coordinates $x = (x^1, \ldots, x^p)$, $u = (u^1, \ldots, u^q)$ and $y = (y^1, \ldots, y^r)$ respectively, that Δ is described by the equations $\{\Delta^l(x, u^{(n)}) = 0 : l = 1, \ldots, m\}$ and that the prolongation is described by

$$0 = \Xi_i^a = y_i^a - F_i^a(x, u^{(n-1)}, y), \quad i = 1, \ldots, p, \quad a = 1, \ldots, r.$$

Assuming that Δ is locally solvable and of maximal rank, a vector field $\mathbf{v}$ on $M \times Y$ generates a symmetry group of (Δ, Ξ) if and only if the functions

$$\begin{aligned} \mathrm{pr}^{(n)}\mathbf{v}(\Delta^l) &: M^{(n)} \times Y \to \mathbb{R}, \quad l = 1, \ldots, m, \\ \mathrm{pr}^{(n-1)}\mathbf{v}(\Xi_i^a) &: M^{(n-1)} \times Y \to \mathbb{R}, \quad a = 1, \ldots, r, \quad i = 1, \ldots, p, \end{aligned} \tag{5.10}$$

are identically zero on $\mathcal{S}_{(\Delta,\Xi)}$, where the prolongations of $\mathbf{v}$ are those described at the end of Section 2.5. This notion of symmetry generators of a Wahlquist-Estabrook prolongation is generalized in the following definition.

Definition 5.5 The vector field $\mathbf{v}$ on $M \times Y$ is called a *partial symmetry generator* of the Wahlquist-Estabrook prolongation (Δ, Ξ) if and only if the functions

$$\mathrm{pr}^{(n)}\mathbf{v}(\Delta^l) : M^{(n)} \times Y \to \mathbb{R}, \quad l = 1, \ldots, m, \tag{5.11}$$

vanish identically on $\mathcal{S}_\Delta \times Y$. If, in addition, $\mathbf{v}$ is not π_1-projectable, where $\pi_1 : M \times Y \to M$ is the trivial projection, it will be known as a *nonlocal partial symmetry generator* of this prolongation. □

A number of comments concerning this definition are in order. Recall that a symmetry generator $\mathbf{v}$ of Δ is a vector field on M such that the functions

$$\mathrm{pr}^{(n)}\mathbf{v}(\Delta^l) : M^{(n)} \to \mathbb{R}, \quad l = 1, \ldots, m,$$

vanish identically on $\mathcal{S}_\Delta$. Regarding $\mathbf{v}$ as a vector field on $M \times Y$, it follows immediately that the functions in equation (5.11) are identically zero on $\mathcal{S}_\Delta \times Y$, so that every symmetry generator of Δ leads to a partial symmetry generator of any Wahlquist-Estabrook prolongation of Δ. Equally obvious is the fact that any vector field on $M \times Y$ which generates a (classical) symmetry group of (Δ, Ξ) must also be a partial symmetry generator of (Δ, Ξ). Thus, the set of partial symmetry generators of a Wahlquist-Estabrook prolongation of a differential equation includes the infinitesimal symmetry generators of the prolonged system as well as those of the original equation. Conversely, if $\mathbf{v}$ is a π_1-projectable partial symmetry generator of (Δ, Ξ) then it is easily shown that $(\pi_1)_*\mathbf{v}$ generates a symmetry group of Δ.

It is helpful to introduce an equivalence relation on the vector space of partial symmetry generators of a prolonged system. If $\mathbf{v}$ is a partial symmetry generator of (Δ, Ξ) then the same is true of $\mathbf{v} + \mathbf{w}$ for any vector field $\mathbf{w}$ on $M \times Y$ such that $(\pi_1)_*\mathbf{w} = 0$. Therefore one can define an equivalence relation on partial symmetry generators by saying that the partial symmetry generators $\mathbf{u}$ and $\mathbf{v}$ are *equivalent* if and only if $(\pi_1)_*(\mathbf{u} - \mathbf{v}) = 0$. Unless stated otherwise, each equivalence class will be represented by the unique member $\mathbf{v}$ satisfying $(\pi_2)_*\mathbf{v} = 0$ where $\pi_2 : M \times Y \to Y$ is the trivial projection.

The next example confirms that Definition 5.5 is a true generalization of the usual symmetries by identifying two partial symmetry generators of a prolongation of the mKdV equation which are not genuine symmetries of either the mKdV equation or its prolongation. In particular, it demonstrates that the determination of partial symmetry generators of a prolongation of a differential equation is only very slightly more difficult than the calculation of (classical) symmetry generators of the original differential equation. The reader should compare the calculations involved in the following example with those of Example 2.44 of [72] which constructed classical symmetries of the KdV equation.

Example 5.6 All partial symmetry generators of the Wahlquist-Estabrook prolongation (Δ, Ξ) of the mKdV equation which featured in Example 5.4 will be constructed here. The process will be treated in some detail to emphasize the similarity to the construction of (classical) symmetries. Take (x, t), v and w as coordinates for $X = \mathbb{R}^2$, $U = \mathbb{R}^1$ and $Y = \mathbb{R}^1$ respectively, and let $M = X \times U$. Everything takes place on the prolonged jet space $M^{(3)} \times Y$ and the appropriate subvariety of $M^{(3)}$ is $\mathcal{S}_\Delta = \ker \Delta$ where

$$\Delta : M^{(3)} \to \mathbb{R}, \quad \Delta = v_t + v_{xxx} - 6v^2 v_x. \tag{5.12}$$

The prolonged total derivative operators corresponding to the Wahlquist-Estabrook prolongation are

$$\tilde{D}_x = D_x + v\partial_w, \quad \tilde{D}_t = D_t + (-v_{xx} + 2v^3)\partial_w.$$

The most general partial symmetry generator of this prolongation has a representative

$$\mathbf{v} = f(x,t,v,w)\partial_x + g(x,t,v,w)\partial_t + h(x,t,v,w)\partial_v \tag{5.13}$$

for appropriate smooth functions f, g, h, and

$$\mathrm{pr}^{(3)}\mathbf{v}(\Delta) = h^t + h^{xxx} - 6v^2 h^x - 12vv_x h$$

where the smooth functions h^x, h^t, h^{xxx} on $M^{(3)} \times Y$ are obtained by prolonging the vector field $\mathbf{v}$ to $\mathrm{pr}^{(3)}\mathbf{v}$. They are given by

$$\begin{aligned}
h^x &= \tilde{D}_x(h) - v_x\tilde{D}_x(f) - v_t\tilde{D}_x(g), \\
h^t &= \tilde{D}_t(h) - v_x\tilde{D}_t(f) - v_t\tilde{D}_t(g), \\
h^{xxx} &= \tilde{D}_x^3(h) - v_x\tilde{D}_x^3(f) - 3v_{xx}\tilde{D}_x^2(f) - 3v_{xxx}\tilde{D}_x(f) \\
&\quad - v_t\tilde{D}_x^3(g) - 3v_{xt}\tilde{D}_x^2(g) - 3v_{xxt}\tilde{D}_x(g),
\end{aligned}$$

where the total derivative operators one usually finds in these expressions have been replaced by the corresponding prolonged total derivative operators. Therefore, the vector field $\mathbf{v}$ of equation (5.13) is a partial symmetry generator of (Δ, Ξ) if and only if

$$h^t + h^{xxx} - 6v^2 h^x - 12vv_x h = 0$$

on $\mathcal{S}_\Delta \times Y$. This equation is solved in Appendix B using an interactive REDUCE session. The general solution to the determining equation has

$$\begin{aligned}
f &= c_2 x + c_3, \\
g &= 3c_1 + 3c_2 t, \\
h &= -c_2 v + c_4 e^{2w} + c_5 e^{-2w},
\end{aligned}$$

for arbitrary constants $c_1, \ldots, c_5$. Consequently, the vector space of partial symmetry generators of (Δ, Ξ) is five-dimensional, with basis

$$\left\{\partial_x,\ x\partial_x + 3t\partial_t - v\partial_v,\ \partial_t,\ e^{2w}\partial_v,\ e^{-2w}\partial_v\right\}.$$

Three vectors are the expected infinitesimal symmetries of the mKdV equation, while two others, $e^{2w}\partial_v$ and $e^{-2w}\partial_v$, are genuinely new. They are examples of nonlocal partial symmetry generators of the prolongation and, from Example 5.4, cannot correspond to true nonlocal symmetries. As claimed, Definition 5.5 is truly a generalization of the usual notion of symmetry generator. □

The preceding example has demonstrated that partial symmetry generators are more widespread than true symmetry generators, but they have one other advantage. Suppose that the Wahlquist-Estabrook prolongation (Δ, Ξ) of Δ has been augmented to a larger prolongation (Δ, Ξ, Υ) on $M \times Y \times Z$. If $\mathbf{v}$ is a partial symmetry generator of (Δ, Ξ) then this vector field, when treated as a vector field on $M \times Y \times Z$ rather than $M \times Y$, meets all the requirements to be a partial symmetry generator of (Δ, Ξ, Υ). Thus, when augmenting a Wahlquist-Estabrook prolongation one retains all partial symmetry generators of the original prolongation and may possibly uncover more from the augmented system. The equivalent result need not hold for true symmetry generators.

With the notation introduced prior to Definition 5.5, suppose that the vector field $\mathbf{v}$ on $M \times Y$ is a true symmetry generator of the prolongation (Δ, Ξ). In particular, equations (5.10) are satisfied by the prolongations of this vector field. Now suppose that the prolongation is augmented to (Δ, Ξ, Υ), a system described on $M \times Y \times Z$ by the equations

$$0 = \Upsilon_i^a = z_i^a - G_i^a(x, u^{(n-1)}, y, z), \quad i = 1, \ldots, p, \quad a = 1, \ldots, s,$$

where $z = (z^1, \ldots, z^s)$ are taken as coordinates on $Z = \mathbb{R}^s$. In order to determine whether or not the symmetry generator $\mathbf{v}$ of (Δ, Ξ) leads to an equivalent symmetry of (Δ, Ξ, Υ), one must search for a symmetry generator $\mathbf{u}$ of the latter prolongation with the property that $(\pi_{12})_* \mathbf{u} = \mathbf{v}$. Here π_{12} denotes the trivial projection $\pi_{12} : M \times Y \times Z \to M \times Y$. In terms of the coordinates used here, this requires

$$\mathbf{u} = \mathbf{v} + \sum_{a=1}^{s} \sigma^a(x, u, y, z) \partial_{z^a},$$

where the smooth functions σ^a must be such that

$$\mathrm{pr}^{(n-1)} \mathbf{u}(\Upsilon_i^a) : M^{(n-1)} \times Y \times Z \to \mathbb{R}, \quad i = 1, \ldots, p, \quad a = 1, \ldots, s,$$

vanish identically on $\mathcal{S}_\Delta \times Y \times Z$. Such functions do not always exist, so that symmetry generators can be lost in the process of augmenting existing prolongations.

An example of this phenomenon is provided by the prolongation of the KdV equation described by equations (4.26). Although the KdV equation admits the Galilean symmetry generator $\mathbf{v} = 12t\partial_x + \partial_u$ one finds that the prolonged system features no symmetry generator of the form $12t\partial_x + \partial_u + \sigma(x, t, u, v)\partial_v$.

This result suggests that even if one is actually interested in finding true nonlocal symmetries, it is best to begin by searching for partial symmetry generators. One can proceed by repeatedly augmenting a prolongation, adding more and more pseudopotentials at each step, and calculating the partial symmetry generators of the final prolongation. Unlike with true symmetry generators, one need not worry about the possibility of losing symmetries in the process. The following example demonstrates this technique, answering (and posing) some questions about the origins of the nonlocal partial symmetry generators discovered in Example 5.6.

Example 5.7 As is the case for the KdV equation, the mKdV equation possesses an infinite family of polynomial conservation laws. Example 5.6 constructed two nonlocal partial symmetry generators of the prolongation of the mKdV equation corresponding to the first of these,

$$0 = (v)_t + (v_{xx} - 2v^3)_x.$$

One wonders whether consideration of the higher order conservation laws would yield more nonlocal partial symmetries. A first step towards answering this question is taken in the current example, which determines nonlocal partial symmetry generators of the prolongation representing the first three polynomial conservation laws. These comprise the first one, given above, together with

$$\begin{aligned} 0 &= (v^2)_t + (2vv_{xx} - v_x^2 - 3v^4)_x, \\ 0 &= (v_x^2 + v^4)_t - (2v_xv_t + v_{xx}^2 - 4v^3v_{xx} + 4v^6)_x. \end{aligned}$$

The differential equation Δ is the mKdV equation (5.12) and the final prolongation (Δ, Ξ) is given by equations (5.9), together with

$$\begin{aligned} p_x &= v^2, \\ p_t &= -2vv_{xx} + v_x^2 + 3v^4, \end{aligned}$$

$$\begin{aligned} q_x &= v_x^2 + v^4, \\ q_t &= 2v_x v_t + v_{xx}^2 - 4v^3 v_{xx} + 4v^6. \end{aligned}$$

A calculation very similar to that in Example 5.6 yields only the partial symmetry generators found there, so that adding these higher conservation laws has not led to any additional symmetry of the system. Of course, it is possible that the infinite prolongation involving all conservation laws may display some extra symmetry, but it might be that the nonlocal symmetries are determined just by the first of this infinite family. Although it is not pursued here, this would seem to be a problem worth investigating. □

Attention now focuses on applications for nonlocal partial symmetry generators. As mentioned previously, the idea is to further prolong a Wahlquist-Estabrook prolongation possessing a nonlocal partial symmetry generator until the augmented system admits a true nonlocal symmetry generator. The process is now described in detail.

Suppose that $\mathbf{u}$ is a partial symmetry generator of (Δ, Ξ) such that $(\pi_2)_* \mathbf{u} = 0$, with $\pi_2 : M \times Y \to Y$ the trivial projection. One might ask whether $\mathbf{u}$ is equivalent to a true symmetry generator of that prolongation. The most general partial symmetry generator in the same equivalence class as $\mathbf{u}$ is

$$\mathbf{v} = \mathbf{u} + \sum_{a=1}^{r} \psi^a(x, u, y) \partial_{y^a}$$

and this vector field generates a symmetry group of (Δ, Ξ) if and only if the appropriate prolongation of $\mathbf{v}$ preserves the prolongation equations

$$0 = \Xi_i^a = y_i^a - F_i^a(x, u^{(n-1)}, y), \quad i = 1, \ldots, p, \quad a = 1, \ldots, r.$$

In terms of local coordinates on $M \times Y$, $\mathbf{u}$ has the form

$$\mathbf{u} = \sum_{i=1}^{p} \xi^i(x, u, y) \partial_{x^i} + \sum_{\alpha=1}^{q} \phi^\alpha(x, u, y) \partial_{u^\alpha}.$$

It follows that the coefficient of $\partial_{y_i^a}$ in the prolongation of $\mathbf{v}$ is the function

$$\psi_i^a = \tilde{D}_{x^i}(\psi^a) - \sum_{j=1}^{p} \tilde{D}_{x^i}(\xi^j) y_j^a, \quad i = 1, \ldots, p, \quad a = 1, \ldots, r,$$

on $M^{(n-1)} \times Y^{(1)}$. The prolongation equations $\Xi^a_j = 0$ allow one to replace y^a_j by F^a_j and write instead

$$\psi^a_i = \tilde{D}_{x^i}(\psi^a) - \sum_{j=1}^{p} \tilde{D}_{x^i}(\xi^j) F^a_j, \quad i = 1, \ldots, p, \quad a = 1, \ldots, r,$$

which is now a function on $M^{(n-1)} \times Y$. Consequently, $\mathbf{v}$ generates a symmetry group of (Δ, Ξ) if and only if

$$\begin{aligned} 0 &= \mathrm{pr}^{(n-1)}\mathbf{v}(\Xi^a_i) \\ &= \psi^a_i - \mathrm{pr}^{(n-1)}\mathbf{v}(F^a_i) \\ 0 &= \tilde{D}_{x^i}(\psi^a) - \sum_{j=1}^{p} \tilde{D}_{x^i}(\xi^j) F^a_j - \mathrm{pr}^{(n-1)}\mathbf{u}(F^a_i) - \sum_{b=1}^{r} \psi^b \frac{\partial F^a_i}{\partial y^b}, \end{aligned} \tag{5.14}$$

on $\mathcal{S}_\Delta \times Y$ for all $i = 1, \ldots, p$ and $a = 1, \ldots, r$.

By replacing the unknown functions ψ^a in equations (5.14) with new dependent variables z^a one obtains an augmented prolongation of (Δ, Ξ). The next proposition proves that a suitably altered version of equations (5.14) satisfies the requirements of a Wahlquist-Estabrook prolongation.

Proposition 5.8 *Let $\Delta[u] = 0$ denote a system of n-th order differential equations on $M \subseteq X \times U$ with Wahlquist-Estabrook prolongation (Δ, Ξ) to $M \times Y$, where $X = \mathbb{R}^p$, $U = \mathbb{R}^q$ and $Y = \mathbb{R}^r$ have coordinates $x = (x^1, \ldots, x^p)$, $u = (u^1, \ldots, u^q)$ and $y = (y^1, \ldots, y^r)$ respectively. Suppose that the prolongation equations are*

$$0 = \Xi^a_i = y^a_i - F^a_i(x, u^{(n-1)}, y), \quad i = 1, \ldots, p, \quad a = 1, \ldots, r.$$

If $\mathbf{u}$ is a partial symmetry generator of (Δ, Ξ) with $(\pi_2)_*\mathbf{u} = 0$ then there exists a Wahlquist-Estabrook prolongation of Δ to (Δ, Ξ, Υ) on $M \times Y \times Z$, where $Z = \mathbb{R}^r$ has coordinates $z = (z^1, \ldots, z^r)$. The additional prolongation equations are

$$0 = \Upsilon^a_i = z^a_i - \sum_{b=1}^{r} z^b \frac{\partial F^a_i}{\partial y^b} - \sum_{j=1}^{p} \tilde{D}_{x^i}(\xi^j) F^a_j - \mathrm{pr}^{(n-1)}\mathbf{u}(F^a_i), \tag{5.15}$$

where $i = 1, \ldots, p$ and $a = 1, \ldots, r$.

PROOF: The prolonged total derivative operators are

$$\tilde{D}_{x^i} = D_{x^i} + \sum_{a=1}^{r} F^a_i \frac{\partial}{\partial y^a} + \sum_{a=1}^{r} G^a_i \frac{\partial}{\partial z^a}, \quad i = 1, \ldots, p,$$

where

$$G_i^a = \sum_{b=1}^{r} z^b \frac{\partial F_i^a}{\partial y^b} + \sum_{j=1}^{p} \tilde{D}_{x^i}(\xi^j) F_j^a + \mathrm{pr}^{(n-1)}\mathbf{u}(F_i^a), \quad i = 1, \ldots, p, \quad a = 1, \ldots, r.$$

It is sufficient to prove that the functions

$$\tilde{D}_{x^i}(G_j^a) - \tilde{D}_{x^j}(G_i^a), \quad i, j = 1, \ldots, p, \quad a = 1, \ldots, r,$$

vanish identically on $\mathcal{S}_\Delta \times Y \times Z$.

This lengthy process involves reasonably straightforward calculations. The intermediate steps, only, are presented here. Notice that

$$\begin{aligned} \tilde{D}_{x^j}(G_i^a) &= \sum_{b=1}^{r} \left(z^b \tilde{D}_{x^j}\left(\frac{\partial F_i^a}{\partial y^b}\right) + G_j^b \frac{\partial F_i^a}{\partial y^b} \right) \\ &\quad + \sum_{k=1}^{p} \left(\tilde{D}_{x^j}(\tilde{D}_{x^i}(\xi^k)) F_k^a + \tilde{D}_{x^i}(\xi^k) \tilde{D}_{x^j}(F_k^a) \right) + \tilde{D}_{x^j}(\mathrm{pr}^{(n-1)}\mathbf{u}(F_i^a)), \end{aligned}$$

and, after a simple calculation, that

$$\tilde{D}_{x^j}\left(\frac{\partial F_i^a}{\partial y^b}\right) = \frac{\partial}{\partial y^b}\left(\tilde{D}_{x^j}(F_i^a)\right) - \sum_{c=1}^{r} \frac{\partial F_i^a}{\partial y^c} \frac{\partial F_j^c}{\partial y^b}. \tag{5.16}$$

The prolongation of $\mathbf{u}$ is

$$\mathrm{pr}^{(n-1)}\mathbf{u} = \sum_{k=1}^{p} \xi^k \partial_{x^k} + \sum_{|J|=0}^{n-1} \sum_{\alpha=1}^{q} \phi_J^\alpha \partial_{u_J^\alpha},$$

where

$$\phi_{Jj}^\alpha = \tilde{D}_{x^j}(\phi_J^\alpha) - \sum_{k=1}^{p} u_{Jk}^\alpha \tilde{D}_{x^j}(\xi^k)$$

and the terms involving $\partial_{y_J^a}$, $\partial_{z_J^a}$ have been omitted as they are not required here. Thus

$$\begin{aligned} \tilde{D}_{x^j}(\mathrm{pr}^{(n-1)}\mathbf{u}(F_i^a)) &= \sum_{k=1}^{p} \left(\tilde{D}_{x^j}(\xi^k) \frac{\partial F_i^a}{\partial x^k} + \xi^k \tilde{D}_{x^j}\left(\frac{\partial F_i^a}{\partial x^k}\right) \right) \\ &\quad + \sum_{|J|=0}^{n-1} \sum_{\alpha=1}^{q} \left(\tilde{D}_{x^j}\left(\phi_J^\alpha\right) \frac{\partial F_i^a}{\partial u_J^\alpha} + \phi_J^\alpha \tilde{D}_{x^j}\left(\frac{\partial F_i^a}{\partial u_J^\alpha}\right) \right) \\ &= \sum_{k=1}^{p} \tilde{D}_{x^j}(\xi^k) D_{x^k}(F_i^a) + \sum_{b=1}^{r} F_j^b \cdot \mathrm{pr}^{(n-1)}\mathbf{u}\left(\frac{\partial F_i^a}{\partial y^b}\right) \\ &\quad + \mathrm{pr}^{(n-1)}\mathbf{u}(D_{x^j}(F_i^a)) + \sum_{|J|=n-1} \sum_{\alpha=1}^{q} \phi_{Jj}^\alpha \frac{\partial F_i^a}{\partial u_J^\alpha}. \end{aligned}$$

Using this expression and equation (5.16), one can write

$$\begin{aligned}\tilde{D}_{x^J}(G_i^a) &= \sum_{k=1}^{p}\left(\tilde{D}_{x^J}(\xi^k)\tilde{D}_{x^k}(F_i^a)+\tilde{D}_{x^J}(\tilde{D}_{x^i}(\xi^k)F_k^a)\right)\\ &\quad + \mathrm{pr}^{(n)}\mathbf{u}(\tilde{D}_{x^J}(F_i^a))+\sum_{b=1}^{r} z^b\frac{\partial}{\partial y^b}\left(\tilde{D}_{x^J}(F_i^a)\right).\end{aligned}$$

The functions

$$\theta_{ij}^a=\tilde{D}_{x^i}(F_j^a)-\tilde{D}_{x^J}(F_i^a),\quad i,j=1,\ldots,p,\ \ a=1,\ldots,r,$$

vanish identically on $\mathcal{S}_\Delta\times Y\times Z$ since (Δ,Ξ) is a Wahlquist-Estabrook prolongation of Δ, and so it follows that

$$\begin{aligned}&\tilde{D}_{x^i}(G_j^a)-\tilde{D}_{x^J}(G_i^a)\\ &= \sum_{b=1}^{r} z^b\frac{\partial\theta_{ij}^a}{\partial y^b}+\mathrm{pr}^{(n)}\mathbf{u}(\theta_{ij}^a)\\ &\quad+\sum_{k=1}^{p}\left(\tilde{D}_{x^J}(\xi^k)\cdot\theta_{ik}^a+\tilde{D}_{x^i}(\xi^k)\cdot\theta_{kj}^a+F_k^a(\tilde{D}_{x^i}\tilde{D}_{x^J}-\tilde{D}_{x^J}\tilde{D}_{x^i})(\xi^k)\right).\end{aligned}$$

Since the functions $\{\theta_{ij}^a : i,j=1,\ldots,p,\ a=1,\ldots,r\}$ vanish identically on $\mathcal{S}_\Delta\times Y\times Z$, it follows that their y^b-derivatives must also vanish on that restricted domain. Furthermore, since $\mathrm{pr}^{(n)}\mathbf{u}$ is tangent to $\mathcal{S}_\Delta\times Y\times Z$, each function $\mathrm{pr}^{(n)}\mathbf{u}(\theta_{ij}^a)$ must vanish there also. Thus each function $\tilde{D}_{x^i}(G_j^a)-\tilde{D}_{x^J}(G_i^a)$ is identically zero on $\mathcal{S}_\Delta\times Y\times Z$. □

Given a differential equation Δ the goal here is to discover a Wahlquist-Estabrook prolongation of Δ which admits a nonlocal symmetry generator. If the prolongation (Δ,Ξ) admits a nonlocal partial symmetry generator $\mathbf{u}$, the obvious first step is to attempt to solve equations (5.14) for smooth functions ψ^a on $M\times Y$ and, if a solution can be found, $\mathbf{u}+\sum_{a=1}^{r}\psi^a\partial_{y^a}$ is a nonlocal symmetry generator. When equations (5.14) cannot be solved, one instead extends (Δ,Ξ) to a new prolongation (Δ,Ξ,Υ) as described in Proposition 5.8. The vector field $\mathbf{u}$ remains a nonlocal partial symmetry generator of this prolongation and the procedure of attempting to prolong it to a true nonlocal symmetry generator of (Δ,Ξ,Υ) is repeated. This time, $\mathbf{u}$ is equivalent to

$$\mathbf{v}=\mathbf{u}+\sum_{a=1}^{r}z^a\partial_{y^a}+\sum_{a=1}^{r}\sigma^a(x,u,y,z)\partial_{z^a},$$

where the substitution $\psi^a \mapsto z^a$ has been made, consistent with equations (5.14) and (5.15). A system of equations for $\{\sigma^a : a = 1, \ldots, r\}$ results from requiring that $\mathrm{pr}^{(n-1)}\mathbf{v}$ preserves the new prolongation equations, $\Upsilon[u, y, z] = 0$. If these equations can be solved for some smooth functions σ^a on $M \times Y \times Z$ then $\mathbf{v}$ is a true symmetry generator, otherwise (Δ, Ξ, Υ) must be prolonged again. This process continues until, hopefully, a prolongation of Δ is found which features a true nonlocal symmetry generator equivalent to $\mathbf{u}$.

Example 5.6 identified two nonlocal partial symmetry generators of a particularly simple prolongation of the mKdV equation. In the following example this prolongation is extended to one which admits a true nonlocal symmetry generator, demonstrating the technique suggested above.

Example 5.9 The vector field $\mathbf{u} = e^{2w}\partial_v$ was shown in Example 5.6 to be a nonlocal partial symmetry generator of the prolongation (Δ, Ξ) of the mKdV equation involving the potential w defined by equations (5.9). Consider the vector field

$$\mathbf{v} = e^{2w}\partial_v + \psi(x, t, v, w)\partial_w$$

which is the most general partial symmetry generator equivalent to $\mathbf{u}$. The prolongation equations (5.9) are invariant under the group action generated by $\mathrm{pr}^{(2)}\mathbf{v}$ if and only if ψ satisfies

$$\tilde{D}_x(\psi) = e^{2w}, \quad \tilde{D}_t(\psi) = -2e^{2w}(v_x - v^2). \tag{5.17}$$

It is easily confirmed that there is no smooth function $\psi(x, t, v, w)$ satisfying these equations so that, as predicted by Example 5.4, $\mathbf{u}$ is not equivalent to a nonlocal symmetry of (Δ, Ξ).

Proposition 5.8 allows one to extend (Δ, Ξ) to a prolongation (Δ, Ξ, Υ) of the mKdV equation by introducing the pseudopotential y defined by

$$y_x = e^{2w}, \quad y_t = 2e^{2w}(-v_x + v^2).$$

The nonlocal partial symmetry generator $\mathbf{u}$ is now equivalent to the vector field

$$\mathbf{v} = e^{2w}\partial_v + \psi(x, t, v, w, y)\partial_w + \sigma(x, t, v, w, y)\partial_y,$$

which determines a nonlocal symmetry of (Δ, Ξ, Υ) if and only if ψ satisfies equations (5.17), where $\tilde{D}_x$ and $\tilde{D}_t$ are replaced by the new prolonged total derivative operators, and σ satisfies

$$\tilde{D}_x(\sigma) = 2e^{2w}\psi, \quad \tilde{D}_t(\sigma) = -4e^{2w}(v_x - v^2)\psi.$$

These equations have general solution

$$\psi = y + c_1, \quad \sigma = y^2 + 2c_1 y + c_2,$$

for arbitrary constants c_1 and c_2. Thus, not only does (Δ, Ξ, Υ) possess a true nonlocal symmetry generator $e^{2w}\partial_v + y\partial_w + y^2\partial_y$ equivalent to $\mathbf{u}$, it also admits a full internal symmetry group which is generated by ∂_y and $\partial_w + 2y\partial_y$. □

Example 5.4 showed that $\mathbf{w} = e^{-2w}\partial_v$ was also a nonlocal partial symmetry generator of the prolongation (Δ, Ξ) of the mKdV equation being considered here. Unlike $\mathbf{u}$, however, $\mathbf{w}$ is not equivalent to a genuine nonlocal symmetry generator of the prolongation (Δ, Ξ, Υ) of the preceding example, so that this prolongation would have to be augmented further for it to admit two nonlocal symmetry generators equivalent to $\mathbf{u}$ and $\mathbf{w}$. It will be seen in Chapter 6 that such prolongations possess infinitely many nonlocal symmetry generators which span a subalgebra of $\mathfrak{sl}(2, \mathbb{R}) \otimes \mathbb{R}[\lambda, \lambda^{-1}]$.

5.3 Equations with recursion operators

Section 5.2 attacked the problem of constructing Wahlquist-Estabrook prolongations of a given differential equation which admit a nonlocal symmetry generator by using partial symmetry generators. For many differential equations, the search for such partial symmetries is far from simple and involves determining the partial symmetry spaces of more and more general Wahlquist-Estabrook prolongations. This section develops a method which can sometimes yield prolongations admitting nonlocal partial symmetry generators for equations possessing a recursion operator.

Recall the definition of a generalized symmetry from Section 2.4. A recursion operator of an n-th order differential equation Δ is usually defined to be a special linear operator $\mathcal{R} : \mathfrak{A}^q \to \mathfrak{A}^q$ on the space of q-tuples of differential functions, where Δ involves q dependent variables. $\mathcal{R}$ must be such that whenever the evolutionary vector field $\mathbf{v}_Q$ is a generalized symmetry of Δ then so is $\mathbf{v}_{Q'}$, with $Q' = \mathcal{R}(Q)$

(see Definition 5.24 of [72]). A well known result states that if there exists a linear differential operator $\mathcal{S} : \mathfrak{A}^m \to \mathfrak{A}^m$ with the property that

$$\mathsf{D}_\Delta \mathcal{R} = \mathcal{S} \mathsf{D}_\Delta \tag{5.18}$$

for all solutions to Δ, then $\mathcal{R}$ is a recursion operator for Δ. Here D_Δ denotes the Fréchet derivative associated with the differential equation

$$\Delta^l(x, u^{(n)}) = 0, \quad l = 1, \ldots, m.$$

It is an $m \times q$ matrix of differential operators with entries

$$(\mathsf{D}_\Delta)^l_\alpha = \sum_{|J|=0}^{n} \frac{\partial \Delta^l}{\partial u^\alpha_J} D_J, \quad l = 1, \ldots, m, \quad \alpha = 1, \ldots, q,$$

where $D_J = D_{x^{j_1}} \cdots D_{x^{j_r}}$ (see Theorem 5.30 of [72]).

With nonlinear differential equations it is often the case that recursion operators are "integro-differential" operators, rather than the more familiar differential operators. This phenomenon can create problems if one is not careful. For example, the mKdV equation

$$0 = \Delta[v] = v_t + v_{xxx} - 6v^2 v_x$$

is usually said to have recursion operator

$$\mathcal{R} = D_x^2 - 4v^2 - 4v_x D_x^{-1} \cdot v \tag{5.19}$$

because the operator

$$\begin{aligned} \mathsf{D}_\Delta \mathcal{R} - \mathcal{R} \mathsf{D}_\Delta &= -8v(v_t + v_{xxx} - 6v^2 v_x) \\ &\quad - 4v_x(v_{xt} + v_{xxxx} - 6v^2 v_{xx} - 12 v v_x^2) D_x^{-1} \cdot v \\ &\quad - 4v_x D_x^{-1} \cdot (v_t + v_{xxx} - 6v^2 v_x) \end{aligned}$$

appears to vanish on solutions to the mKdV equation. In fact, for any differential function $Q \in \mathfrak{A}$ all one can say is that

$$(\mathsf{D}_\Delta \mathcal{R} - \mathcal{R} \mathsf{D}_\Delta)(Q) = -4v_x D_x^{-1}(0)$$

on solutions of the mKdV equation. Consequently,

$$(\mathsf{D}_\Delta \mathcal{R} - \mathcal{R} \mathsf{D}_\Delta)(Q) = v_x h(t)$$

for an arbitrary smooth function h, so that $\mathcal{R}$ does not satisfy the criteria for a recursion operator at all! The reason is that the *differential* operator $\mathcal{S}$ in equation (5.18) has been replaced by the *integro-differential* operator $\mathcal{R}$. One is led to ask the question: Is $\mathcal{R}$, given by equation (5.19), a recursion operator for the mKdV equation? An answer is contained in the following example.

Example 5.10 The x-translational symmetry of the mKdV equation has characteristic $Q = v_x$. In order to evaluate $\mathcal{R}(Q)$ one must first identify the most general function $P \in \mathfrak{A}$ satisfying $D_x(P) = vQ$. Clearly, $P = \frac{1}{2}v^2 + f(t)$ for an arbitrary smooth function f, so that

$$\begin{aligned} Q' = \mathcal{R}(Q) &= D_x^2(Q) - 4v^2Q - 4v_xD_x^{-1}(vQ) \\ &= D_x^2(Q) - 4v^2Q - 4v_xD_x^{-1}(D_x(P)) \\ &= v_{xxx} - 6v^2v_x - 4v_xf(t). \end{aligned}$$

On solutions of the mKdV equation, $Q' = -v_t - 4v_xf(t)$, which is the characteristic of the vector field $\mathbf{v} = \partial_t + 4f(t)\partial_x$. It is well known that the only such vector field being a symmetry of the mKdV equation has f independent of t. Thus, although Q is the characteristic of a symmetry of the mKdV equation, $Q' = \mathcal{R}(Q)$ does not, in general, determine such a symmetry.

One is entitled to think that this is merely being pedantic — after all, surely the restriction $f(t) = a$ with a an arbitrary constant is a natural one? Example 5.11 will show that such "natural" choices are not always this simple. One way to avoid this problem is to replace the ambiguous term, $D_x^{-1}(Q)$ in this case, by a system of equations determining any arbitrary functions to the required precision. It is certainly possible to do this in the case of the mKdV equation. $Q \in \mathfrak{A}$ determines a generalized symmetry $\mathbf{v}_Q$ of the mKdV equation if and only if

$$0 = \mathsf{D}_\Delta(Q) = (D_t + D_x^3 - 6v^2D_x - 12vv_x)(Q)$$

on solutions of Δ. For such a differential function, the system of equations

$$D_x(P) = vQ, \quad D_t(P) = (-vD_x^2 + v_xD_x + (-v_{xx} + 6v^3))(Q), \tag{5.20}$$

is integrable, in the sense that

$$(D_tD_x - D_xD_t)(P) = D_t(vQ) - D_x(-vD_x^2 + v_xD_x + (-v_{xx} + 6v^3))(Q)$$

$$= v\mathsf{D}_\Delta(Q)$$
$$= 0$$

on solutions to the mKdV equation. The solution P to the system of equations (5.20), if it exists, is determined up to an additive constant. Noting that $Q = v^{-1}D_x(P)$ one can define the new differential function

$$Q' = \mathcal{R}(Q) = \mathcal{R} \cdot v^{-1}D_x(P)$$

and a straightforward calculation confirms that

$$Q' = D_x \cdot (D_x \cdot v^{-1}D_x - 4v)(P).$$

Furthermore, using equations (5.20) it is easy to show that $\mathsf{D}_\Delta(Q') = 0$, which implies that $\mathbf{v}_{Q'}$ is a generalized symmetry of the mKdV equation.

In the special case where $Q = v_x$, equations (5.20) force $P = \frac{1}{2}v^2 + c$ for an arbitrary constant c, and $Q' = -v_t - 4cv_x$. This time Q' does yield a symmetry of the mKdV equation. □

A straightforward change of coordinates for the mKdV equation results in the undesirable behaviour hinted at in Example 5.10.

Example 5.11 The invertible change of coordinates $(x, t, v) \mapsto (y, s, v)$ given by $y = xt$ and $s = t$ gives the mKdV equation in the form

$$0 = sv_s + s^4 v_{yyy} - 6s^2v^2v_y + yv_y \tag{5.21}$$

and the recursion operator of equation (5.19) in the form

$$\mathcal{R} = s^2D_y^2 - 4v^2 - 4v_yD_y^{-1} \cdot v.$$

In these coordinates the x-translational symmetry group is generated by $-s\partial_y$, a vector field with characteristic $Q = sv_y$. Simple calculations show that

$$Q' = \mathcal{R}(Q) = -v_s - s^{-1}yv_y - 4g(s)v_y$$

with g an arbitrary smooth function introduced when evaluating D_y^{-1}. The vector field $\partial_s + s^{-1}y\partial_y + 4g(s)\partial_y$ is a symmetry group of equation (5.21) if and only if

$g(s) = as$ for an arbitrary constant a — not the "natural" choice $g(s) = a$ which one might expect. This occurs due to the (perfectly reasonable) simplification

$$v_x D_x^{-1} \cdot v = s v_y \cdot (D_y^{-1} \cdot s^{-1}) \cdot v = v_y D_y^{-1} \cdot v$$

made when expressing $\mathcal{R}$ in the new coordinates. □

Examples 5.10 and 5.11 illustrate some of the difficulties which can arise from integro-differential recursion operators. The following definition is motivated by Example 5.10 and yields recursion operators without the ambiguities mentioned above. It provides an, admittedly cumbersome, way around the difficulties associated with integro-differential operators. This definition also has the advantage that it treats the usual types of recursion operators and their inverses on an equal footing, which is very useful when searching for prolongations of a differential equation which admit nonlocal partial symmetry generators. It does this by replacing every appearance of $D_{x^i}^{-1}$ in the usual recursion operator by a system of first order differential equations.

Definition 5.12 Let Δ denote a system of differential equations on $M \subseteq X \times U$ where $X = \mathbb{R}^p$ and $U = \mathbb{R}^q$. A *recursion operator* for Δ comprises the matrices of elements of $\mathfrak{A}$ (with dimensions in brackets) $\{\mathcal{A}_i(s \times s) : i = 1, \ldots, p\}$ together with the matrices of differential operators on $\mathfrak{A}$ $\{\mathcal{B}_i(s \times q), \mathcal{C}(q \times s), \mathcal{D}(q \times q) : i = 1, \ldots, p\}$, where s is some nonnegative integer, provided that they satisfy

$$\begin{aligned} D_{x^i}(\mathcal{A}_j) - D_{x^j}(\mathcal{A}_i) + \mathcal{A}_j \mathcal{A}_i - \mathcal{A}_i \mathcal{A}_j &= 0, \\ D_{x^i} \mathcal{B}_j - D_{x^j} \mathcal{B}_i - \mathcal{A}_i \mathcal{B}_j + \mathcal{A}_j \mathcal{B}_i &= \sum_{|J|=0}^{\infty} f_{ij}^J D_J \cdot \mathsf{D}_\Delta, \end{aligned} \tag{5.22}$$

modulo $\Delta[u] = 0$, for suitable differential functions $f_{ij}^J \in \mathfrak{A}$, and that

$$\mathsf{D}_\Delta \left(\mathcal{C}(P) + \mathcal{D}(Q) \right) = 0 \tag{5.23}$$

for all differential functions $P \in \mathfrak{A}^s$ and $Q \in \mathfrak{A}^q$, modulo the equations

$$D_{x^i}(P) = \mathcal{A}_i P + \mathcal{B}_i(Q), \quad i = 1, \ldots, p,$$

$\Delta[u] = 0$ and $\mathsf{D}_\Delta(Q) = 0$. □

As suggested by Example 5.10, the differential operators

$$\begin{aligned}
\mathcal{A}_x &= 0, \\
\mathcal{A}_t &= 0, \\
\mathcal{B}_x &= v, \\
\mathcal{B}_t &= -vD_x^2 + v_x D_x + (-v_{xx} + 6v^3), \\
\mathcal{C} &= D_x \cdot (D_x \cdot v^{-1} D_x - 4v), \\
\mathcal{D} &= 0,
\end{aligned}$$

comprise a recursion operator for the mKdV equation.

As one would hope, these new recursion operators can be used to generate new generalized symmetries of systems of differential equations.

Theorem 5.13 *Suppose that the differential operators* $\{\mathcal{A}_i, \mathcal{B}_i, \mathcal{C}, \mathcal{D} : i = 1, \ldots, p\}$ *form a recursion operator for the system of differential equations* Δ *and that* $\mathbf{v}_Q$ *is a generalized symmetry for some* $Q \in \mathfrak{A}^q$. *Then the system of equations*

$$D_{x^i}(P) = \mathcal{A}_i P + \mathcal{B}_i(Q), \quad i = 1, \ldots, p,$$

is integrable, in the sense that

$$D_{x^i} D_{x^j}(P) = D_{x^j} D_{x^i}(P), \quad i, j = 1, \ldots, p,$$

and for every solution $P \in \mathfrak{A}^s$ *of this system, the differential function*

$$Q' = \mathcal{C}(P) + \mathcal{D}(Q) \in \mathfrak{A}^q$$

determines a generalized symmetry $\mathbf{v}_{Q'}$ *of* Δ.

PROOF: The integrability property follows directly from equations (5.22) and the observation that Q' determines a generalized symmetry is a consequence of equation (5.23) implying that $\mathsf{D}_\Delta(Q') = 0$. □

Generalized symmetries have themselves been generalized further by allowing their characteristics to be functions of not only the jet variables $(x, u^{(n)})$, for arbitrary finite n, but also of any pseudopotential of the differential equation being studied [54]. These objects have many names, "nonlocal Lie-Bäcklund symmetries" being a common one [52]. To maintain consistency with Definition 5.5, different terminology is used here.

Definition 5.14 Let $\Delta[u] = 0$ denote a system of n-th order differential equations

$$\Delta^l(x, u^{(n)}) = 0, \quad l = 1, \ldots, m,$$

on $M \subseteq X \times U$ with Wahlquist-Estabrook prolongation (Δ, Ξ) to $M \times Y$. The generalized vector field

$$\mathbf{v} = \sum_{i=1}^{p} \xi^i \partial_{x^i} + \sum_{\alpha=1}^{q} \phi^\alpha \partial_{u^\alpha} \tag{5.24}$$

is a *generalized partial symmetry generator* of (Δ, Ξ) if and only if

$$\mathrm{pr}^{(n)}\mathbf{v}(\Delta^l) = 0, \quad l = 1, \ldots, m,$$

on solutions to (Δ, Ξ). Here, the functions $\{\xi^i, \phi^\alpha : i = 1, \ldots, p, \ \alpha = 1, \ldots, q\}$ depend on $(x, u^{(s)}, y)$, for arbitrary finite s. If at least one of these functions depends nontrivially on a pseudopotential then $\mathbf{v}$ is called a *generalized nonlocal partial symmetry generator* of (Δ, Ξ). □

The recursion operators introduced in Definition 5.12 are readily generalized to yield a technique capable of mapping generalized partial symmetry generators into new generalized partial symmetry generators. Some notation is introduced before defining these new recursion operators. If the Wahlquist-Estabrook prolongation (Δ, Ξ) is defined on $M \times Y$ with $M \subseteq X \times U$, let $\tilde{\mathfrak{A}}$ denote the algebra of functions of $(x, u^{(n)}, y)$, for arbitrary finite n. Thus, $\tilde{\mathfrak{A}}$ is in some sense the prolongation of $\mathfrak{A}$. The total derivative operators $D_{x^i} : \mathfrak{A} \to \mathfrak{A}$ can be prolonged to $\tilde{D}_{x^i} : \tilde{\mathfrak{A}} \to \tilde{\mathfrak{A}}$ by

$$\tilde{D}_{x^i} = D_{x^i} + \sum_{a=1}^{r} F_i^a \partial_{y^a}, \quad i = 1, \ldots, p,$$

where the prolongation equations are

$$y_i^a = F_i^a(x, u^{(n-1)}, y), \quad i = 1, \ldots, p, \ \ a = 1, \ldots, r,$$

allowing one to prolong an arbitrary differential operator $\mathcal{A} : \mathfrak{A}^q \to \mathfrak{A}^q$ to $\tilde{\mathcal{A}} : \tilde{\mathfrak{A}}^q \to \tilde{\mathfrak{A}}^q$ by replacing every appearance of D_{x^i} by $\tilde{D}_{x^i}$.

Definition 5.15 Let Δ denote a system of differential equations on $M \subseteq X \times U$, where $X = \mathbb{R}^p$ and $U = \mathbb{R}^q$, with Wahlquist-Estabrook prolongation (Δ, Ξ) to $M \times Y$. A *weak recursion operator* for (Δ, Ξ) comprises the matrices of elements of

$\tilde{\mathfrak{A}}$ (with dimensions in brackets) $\{\mathcal{A}_i(s\times s) : i = 1,\ldots,p\}$ together with the matrices of differential operators on $\tilde{\mathfrak{A}}$ $\{\mathcal{B}_i(s \times q), \mathcal{C}(q \times s), \mathcal{D}(q \times q) : i = 1,\ldots,p\}$, where s is some nonnegative integer, provided that they satisfy

$$\begin{aligned} \tilde{D}_{x^i}(\mathcal{A}_j) - \tilde{D}_{x^j}(\mathcal{A}_i) + \mathcal{A}_j\mathcal{A}_i - \mathcal{A}_i\mathcal{A}_j &= 0, \\ \tilde{D}_{x^i}\mathcal{B}_j - \tilde{D}_{x^j}\mathcal{B}_i - \mathcal{A}_i\mathcal{B}_j + \mathcal{A}_j\mathcal{B}_i &= \sum_{|J|=0}^{\infty} f_{ij}^J \tilde{D}_J \cdot \tilde{\mathsf{D}}_\Delta, \end{aligned} \tag{5.25}$$

modulo $\Delta[u] = 0$, $\Xi[u,y] = 0$, for suitable differential functions $f_{ij}^J \in \tilde{\mathfrak{A}}$, and that

$$\tilde{\mathsf{D}}_\Delta(\mathcal{C}(P) + \mathcal{D}(Q)) = 0 \tag{5.26}$$

for all differential functions $P \in \tilde{\mathfrak{A}}^s$ and $Q \in \tilde{\mathfrak{A}}^q$, modulo the equations

$$\tilde{D}_{x^i}(P) = \mathcal{A}_i P + \mathcal{B}_i(Q), \quad i = 1,\ldots,p,$$

$\Delta[u] = 0$, $\Xi[u,y] = 0$ and $\tilde{\mathsf{D}}_\Delta(Q) = 0$. Here $\tilde{\mathsf{D}}_\Delta$ denotes the prolongation of D_Δ. □

Notice the relationship between Definitions 5.12 and 5.15. In particular, every recursion operator $\{\mathcal{A}_i, \mathcal{B}_i, \mathcal{C}, \mathcal{D} : i = 1,\ldots,p\}$ of Δ prolongs to a weak recursion operator $\{\tilde{\mathcal{A}}_i, \tilde{\mathcal{B}}_i, \tilde{\mathcal{C}}, \tilde{\mathcal{D}} : i = 1,\ldots,p\}$ of any prolongation (Δ, Ξ) of Δ. Simply replace every appearance of D_{x^i} by $\tilde{D}_{x^i}$ in the differential operators. For example,

$$\begin{aligned} \tilde{\mathcal{A}}_x &= 0, \\ \tilde{\mathcal{A}}_t &= 0, \\ \tilde{\mathcal{B}}_x &= v, \\ \tilde{\mathcal{B}}_t &= -v\tilde{D}_x^2 + v_x\tilde{D}_x + (-v_{xx} + 6v^3), \\ \tilde{\mathcal{C}} &= \tilde{D}_x \cdot (\tilde{D}_x \cdot v^{-1}\tilde{D}_x - 4v), \\ \tilde{\mathcal{D}} &= 0, \end{aligned}$$

form a weak recursion operator for any prolongation of the mKdV equation.

The converse result is not true — not every weak recursion operator is the prolongation of an ordinary recursion operator. One such weak recursion operator is constructed in the following example.

Example 5.16 A genuinely weak recursion operator for the mKdV equation is constructed here, essentially by inverting the recursion operator of Example 5.10 which was described formally directly after Definition 5.12. When inverting a recursion operator it is sometimes helpful to ignore the refinements introduced earlier in this section and treat the recursion operator as an integro-differential operator in the traditional manner. For the recursion operator of the mKdV equation, if $\mathcal{R} : Q \mapsto Q'$ then $Q' = D_x\mathcal{E}(P)$, where P satisfies equations (5.20) and

$$\mathcal{E} = D_x \cdot v^{-1} D_x - 4v.$$

Since $D_x(P) = vQ$, one could write

$$Q' = D_x \mathcal{E} D_x^{-1} \cdot v(Q)$$

and say that $\mathcal{R} = D_x\mathcal{E}D_x^{-1} \cdot v$. Consequently, if $\mathcal{S}$ is to denote the inverse of $\mathcal{R}$ and $\mathcal{S} : Q' \mapsto Q$ then

$$Q = \mathcal{S}(Q') = v^{-1} D_x \mathcal{E}^{-1} D_x^{-1}(Q').$$

The first difficulty in evaluating Q is the need to determine $D_x^{-1}(Q') = F$. Following the discussion above, one replaces this with a system of equations for F. Clearly, F must satisfy

$$D_x(F) = Q' \tag{5.27}$$

and, since

$$0 = \mathsf{D}_\Delta(Q') = (D_t + D_x^3 - 6v^2 D_x - 12vv_x)(Q')$$

whenever Q' determines a symmetry of the mKdV equation,

$$\begin{aligned} D_t(F) &= D_x^{-1} D_t(Q') \\ &= D_x^{-1}(-D_x^3 + 6v^2 D_x + 12vv_x)(Q') \\ D_t(F) &= (-D_x^2 + 6v^2)(Q'). \end{aligned} \tag{5.28}$$

As required, the system of equations (5.27) and (5.28) is integrable whenever $\mathbf{v}_{Q'}$ is a generalized symmetry of the mKdV equation.

It now follows that

$$Q = v^{-1} D_x \mathcal{E}^{-1}(F)$$

and so the next step is to construct a function E such that

$$(v^{-1}D_x^2 - v^{-2}v_x D_x - 4v)(E) = \mathcal{E}(E) = F. \tag{5.29}$$

If one ignores, temporarily, the t-dependence this is simply a nonhomogeneous, linear, second order ordinary differential equation for E. Equation (5.29) gives

$$((v^{-1}D_x)\cdot(v^{-1}D_x)-4)(E)=v^{-1}F$$

and so $\mathcal{E}(e^{2w})=\mathcal{E}(e^{-2w})=0$ where w is the potential of the mKdV equation described by

$$w_x=v,\quad w_t=-v_{xx}+2v^3.$$

An elementary calculation verifies that

$$E=-\tfrac{1}{4}e^{-2w}G+\tfrac{1}{4}e^{2w}H$$

where G and H must satisfy

$$\tilde{D}_x(G)=e^{2w}F,\quad \tilde{D}_x(H)=e^{-2w}F. \tag{5.30}$$

Just as with the derivation of equation (5.28), straightforward calculations yield the requirements

$$\begin{aligned}\tilde{D}_t(G) &= e^{2w}(-Q'_x+2vQ'-2(v_x-v^2)F),\\ \tilde{D}_t(H) &= e^{-2w}(-Q'_x-2vQ'+2(v_x+v^2)F).\end{aligned} \tag{5.31}$$

Cross-differentiation confirms that the system of equations (5.30) and (5.31) is integrable.

Finally,

$$Q=v^{-1}D_x(E)=v^{-1}D_x(-\tfrac{1}{4}e^{-2w}G+\tfrac{1}{4}e^{2w}H)=\tfrac{1}{2}e^{-2w}G+\tfrac{1}{2}e^{2w}H, \tag{5.32}$$

and one finds that $\tilde{\mathsf{D}}_\Delta(Q)=0$ whenever $\tilde{\mathsf{D}}_\Delta(Q')=0$ and all of equations (5.27), (5.28), and (5.30) to (5.32) are satisfied. In terms of Definition 5.15, one can check that the differential operators

$$\mathcal{A}_x = \begin{pmatrix} 0 & 0 & 0\\ e^{2w} & 0 & 0\\ e^{-2w} & 0 & 0\end{pmatrix},$$

$$\mathcal{A}_t = \begin{pmatrix} 0 & 0 & 0\\ -2e^{2w}(v_x-v^2) & 0 & 0\\ 2e^{-2w}(v_x+v^2) & 0 & 0\end{pmatrix},$$

$$
\begin{aligned}
\mathcal{B}_x &= \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \\
\mathcal{B}_t &= \begin{pmatrix} -\tilde{D}_x^2 + 6v^2 \\ -e^{2w}(\tilde{D}_x - 2v) \\ -e^{-2w}(\tilde{D}_x + 2v) \end{pmatrix}, \\
\mathcal{C} &= \begin{pmatrix} 0 & \frac{1}{2}e^{-2w} & \frac{1}{2}e^{2w} \end{pmatrix}, \\
\mathcal{D} &= 0,
\end{aligned}
$$

form a genuinely weak recursion operator for any prolongation of the mKdV equation involving the potential w. The appearance of w in the components of $\mathcal{A}_x$ is sufficient to prove that this weak recursion operator cannot possibly be the prolongation of a recursion operator as defined in Definition 5.12.

These matrices of differential operators make matters look much more complicated than they really are. To apply this weak recursion operator to the characteristic Q' of a partial symmetry generator of the mKdV equation, simply solve equations (5.27) and (5.28) for F, then solve equations (5.30) and (5.31) for G and H. The new partial symmetry generator has characteristic Q given by equation (5.32). One can verify that the application of the usual recursion operator, described in Example 5.10, to Q gives Q' back again, modulo a term $-4av_x$ which arises from the arbitrary constant introduced when solving equations (5.20). For this reason, the mapping $\mathcal{S} : Q' \mapsto Q$ introduced in this example will be referred to as an application of the inverse of the usual recursion operator for the mKdV equation. □

As claimed above, weak recursion operators map generalized partial symmetry generators into generalized partial symmetry generators.

Theorem 5.17 *Suppose that the prolonged differential operators $\{\mathcal{A}_i, \mathcal{B}_i, \mathcal{C}, \mathcal{D} : i = 1, \ldots, p\}$ form a weak recursion operator for the Wahlquist-Estabrook prolongation (Δ, Ξ) of Δ described on $M \times Y$, where $M \subseteq X \times U$ with $X = \mathbb{R}^p$ and $U = \mathbb{R}^q$. Let $\mathbf{v}_Q$ be a generalized partial symmetry generator of (Δ, Ξ). Then the system of equations*

$$\tilde{D}_{x^i}(P) = \mathcal{A}_i P + \mathcal{B}_i(Q), \quad i = 1, \ldots, p, \tag{5.33}$$

is integrable, in the sense that

$$\tilde{D}_{x^i}\tilde{D}_{x^j}(P) = \tilde{D}_{x^j}\tilde{D}_{x^i}(P), \quad i,j = 1,\ldots,p,$$

and for every solution $P \in \tilde{\mathfrak{A}}^s$ of this system, the differential function

$$Q' = \mathcal{C}(P) + \mathcal{D}(Q) \in \tilde{\mathfrak{A}}^q$$

determines a generalized partial symmetry generator $\mathbf{v}_{Q'}$ of (Δ, Ξ).

PROOF: The proof is similar to that of Theorem 5.13. One finds that $Q \in \tilde{\mathfrak{A}}^q$ determines a generalized partial symmetry generator $\mathbf{v}_Q$ of (Δ, Ξ) if and only if $\tilde{\mathsf{D}}_\Delta(Q) = 0$ on solutions to (Δ, Ξ). Integrability of equations (5.33) is guaranteed by equations (5.25) and the fact that $\mathbf{v}_{Q'}$ is a generalized partial symmetry follows from equation (5.26) implying that $\tilde{\mathsf{D}}_\Delta(Q') = 0$ on solutions of (Δ, Ξ). □

Notice that the new characteristic Q' occurs only when a solution $P \in \tilde{\mathfrak{A}}^s$ can be found to the system of equations (5.33). An especially useful feature of weak recursion operators is that in the case when such a solution does not exist, the situation can be recovered. Suppose that $Q \in \tilde{\mathfrak{A}}^q$ is the characteristic of a generalized partial symmetry generator of (Δ, Ξ), but that equations (5.33) cannot be solved for any function $P \in \tilde{\mathfrak{A}}^s$. One can augment the prolongation (Δ, Ξ) of Δ to a prolongation (Δ, Ξ, Υ) on $M \times Y \times Z$ by introducing new pseudopotentials $\{z^1, \ldots, z^s\}$ defined by the equations

$$z^a_i = G^a_i = \sum_{b=1}^{s} (\mathcal{A}_i)^a_b\, z^b + \sum_{\alpha=1}^{q} (\mathcal{B}_i)^a_\alpha\, (Q^\alpha), \quad i = 1,\ldots,p, \ \ a = 1,\ldots,s.$$

Theorem 5.17 confirms that these equations define a Wahlquist-Estabrook prolongation, as equations (5.25) imply that the functions

$$\tilde{D}_{x^j}(G^a_i) - \tilde{D}_{x^i}(G^a_j), \quad i,j = 1,\ldots,p, \ \ a = 1,\ldots,s,$$

vanish on $\mathcal{S}_\Delta \times Y \times Z$, where

$$\tilde{D}_{x^i} = D_{x^i} + \sum_{a=1}^{r} F^a_i \frac{\partial}{\partial y^a} + \sum_{b=1}^{s} G^b_i \frac{\partial}{\partial z^b}, \quad i = 1,\ldots,p,$$

are the new prolonged total derivative operators. Furthermore, the differential operators $\{\mathcal{A}_i, \mathcal{B}_i, \mathcal{C}, \mathcal{D} : i = 1,\ldots,p\}$ prolong to a weak recursion operator of (Δ, Ξ, Υ)

by replacing the old prolonged total derivative operators with these new ones. Q remains the characteristic of a generalized partial symmetry generator of (Δ, Ξ, Υ), the system of equations (5.33) now have the solution $P = (z^1, \ldots, z^s)^T$ and a new generalized partial symmetry generator of (Δ, Ξ, Υ) has characteristic

$$Q' = \mathcal{C}(P) + \mathcal{D}(Q).$$

The remainder of this section is devoted to using the weak recursion operators defined here to construct Wahlquist-Estabrook prolongations of a given differential equation admitting nonlocal partial symmetry generators. It is important to emphasize here that the symmetry generators being sought are "geometric" vector fields as opposed to "generalized" ones. That is, the functions $\{\xi^i, \phi^\alpha : i = 1, \ldots, p,\ \alpha = 1, \ldots, q\}$ appearing in the generalized vector field in equation (5.24) must depend only on x, u, y and not higher jet variables. Unfortunately, this means that many of the usual recursion operators are useless. Although many of them are integro-differential operators they act in much the same way as pure differential operators by adding higher order jet variables to the symmetry characteristic. One need only refer back to equation (5.19), the usual recursion operator for the mKdV equation, for an example of this behaviour.

One possibility, suggested by Example 5.16, is to try inverses of the usual recursion operators instead. These will act more as integral operators, adding pseudopotentials to symmetry characteristics — but are they necessarily weak recursion operators at all? When the differential equation in question admits a bi-Hamiltonian formulation, the answer is probably yes. (See [72] for details of the Hamiltonian formalism for evolution equations.) The reason for this claim is that if Δ can be described as a Hamiltonian system using two different Hamiltonian operators $\mathcal{D}$ and $\mathcal{E}$ then it follows that $\mathcal{R} = \mathcal{D}\mathcal{E}^{-1}$ is a recursion operator for Δ (Theorem 7.27 of [72]). Then, surely $\mathcal{R}^{-1} = \mathcal{E}\mathcal{D}^{-1}$ must also be a recursion operator for that equation! The reason why the answer to the question above is only "probably" yes is that use of the Hamiltonian formalism is being avoided in this work and, of course, the definition of recursion operator has been changed from [72]. This means that any set of differential operators found by inverting a known recursion operator cannot be assumed to satisfy Definition 5.15 — the requirements of that definition must be checked on a case-by-case basis. Another point to note is that inverting recursion operators can be a difficult process. One usually depends on knowledge of a partial factorization of the recursion operator in question for a successful inversion. Section 6.2 includes a demonstration of this process.

The concept of inverting recursion operators is not new, although this process does not appear to have been used to construct nonlocal symmetries before. Instead, the usual hierarchy of higher order differential equations one can construct with a recursion operator of a differential equation [71] has been extended in the "negative" direction, essentially by applying the inverses of those recursion operators [70], [88].

In order to use recursion operators and their weak counterparts one must have a seed symmetry to which these operators can be applied. It is usual to begin with one of the (classical) symmetry generators of the differential equation being studied, but one possibility which is often overlooked is to begin with the trivial symmetry — that having zero characteristic. This approach is especially productive when used with the refinement of recursion operators introduced earlier in the current section. Even with the usual recursion operators interesting results are obtained.

For instance, return to Example 5.10 and take $Q = 0$ as the characteristic of a symmetry of the mKdV equation. The first step in constructing Q' is to solve

$$D_x(P) = D_t(P) = 0,$$

which clearly has general solution $P = c$ for an arbitrary constant c. It follows that

$$Q' = D_x \cdot (D_x \cdot v^{-1} D_x - 4v)(c) = -4cv_x.$$

One will find that continuing to apply the recursion operator in this way yields the well known family of generalized symmetries of the mKdV equation. Consequently, the action of the usual recursion operator of the mKdV equation on zero yields an infinite-dimensional Abelian symmetry algebra (Theorem 5.20 of [72] confirms the Abelian nature of this Lie algebra). It will be shown in Section 6.3 that the weak recursion operator of Example 5.16, which amounted to the inverse of the usual one, acts on zero to generate another infinite-dimensional Lie algebra — this one is non-Abelian and is a subalgebra of $\mathfrak{sl}(2,\mathbb{R}) \otimes \mathbb{R}[\lambda, \lambda^{-1}]$. The following example constructs the first few partial symmetry generators of this algebra.

Example 5.18 To apply the inverse of the usual recursion operator of the mKdV equation to the trivial symmetry one must first solve equations (5.27) and (5.28) with $Q = 0$. That is,

$$\tilde{D}_x(F) = \tilde{D}_t(F) = 0,$$

so take $F = c_1$ for an arbitrary constant c_1. Then G and H are determined by equations (5.30) and (5.31), which in this case are

$$\begin{aligned}
\tilde{D}_x(G) &= c_1 e^{2w}, \\
\tilde{D}_t(G) &= -2c_1(v_x - v^2)e^{2w}, \\
\tilde{D}_x(H) &= c_1 e^{-2w}, \\
\tilde{D}_t(H) &= 2c_1(v_x + v^2)e^{-2w},
\end{aligned}$$

and it is easily shown that there are no smooth functions $G(x,t,v,w)$, $H(x,t,v,w)$ satisfying these equations. Following the discussion after Theorem 5.17, it is possible to introduce new pseudopotentials of the mKdV equation which will yield a complete solution to the above system. In this case, the pseudopotentials y and z are defined by

$$\begin{aligned}
y_x &= e^{2w}, \\
y_t &= 2e^{2w}(-v_x + v^2), \\
z_x &= e^{-2w}, \\
z_t &= 2e^{-2w}(v_x + v^2).
\end{aligned}$$

Then $G = c_1 y + c_2$ and $H = c_1 z + c_3$ give the general solution and the new partial symmetry generator of the mKdV equation has characteristic

$$Q' = \tfrac{1}{2}c_1(ye^{-2w} + ze^{2w}) + \tfrac{1}{2}c_2(e^{-2w}) + \tfrac{1}{2}c_3(e^{2w}).$$

This recovers the two nonlocal partial symmetry generators of the mKdV equation which were constructed in Example 5.6, as well as deriving a new nonlocal partial symmetry generator $(ye^{-2w} + ze^{2w})\partial_v$. □

5.4 Constructing M-projections: the solution

This section reviews the procedure for constructing M-projections of a given differential equation. It involves solving Problem B — that is, constructing nondegenerate Wahlquist-Estabrook prolongations of the differential equation which admit both a full internal symmetry group and a nonlocal symmetry generator. One begins by searching for prolongations which feature a nonlocal partial symmetry generator. Although this process is not simple, it is considerably easier than looking for prolongations possessing a true nonlocal symmetry generator. When the differential equation admits a recursion operator this first step can often be simplified considerably using the techniques of Section 5.3. Otherwise, the problem can be attacked in an *ad hoc* manner by identifying the partial symmetry generators of various Wahlquist-Estabrook prolongations of the differential equation in question until, if at all, one finds a prolongation which admits a *nonlocal* partial symmetry generator. For the mKdV equation, this step was demonstrated in Example 5.6.

When one has discovered such a prolongation of the differential equation, Proposition 5.8 can be applied repeatedly until, if at all, the initial prolongation is extended to one featuring a genuine nonlocal symmetry generator. Example 5.9 performed this calculation for the prolongation of the mKdV equation appearing in Example 5.6.

At this stage there are two possibilities.

1. The prolongation admits a full internal symmetry group. Any redundant pseudopotentials can be removed using the technique described in Section 2.5 and then an M-projection can be constructed immediately using Theorem 5.2 and the discussion following Problem A. This was done in Example 5.3, showing that the KdV equation arises as an M-projection of the mKdV equation.

2. The prolongation does not admit a full internal symmetry group. This situation is treated in detail in Section 5.5.

All the constructions developed in this chapter have been motivated by properties of the mKdV equation. The following example applies these techniques to another differential equation of some interest, indicating that the approach suggested here will be useful in a general setting.

Example 5.19 Calogero and Degasperis [7] are usually credited with introducing the differential equation

$$p_{t'} = 2p_{x'x'x'} - p_{x'}^3 - 3(c_0^2 e^{-2p} + c_1^2 e^{2p})p_{x'}$$

for $p(x', t')$ where $c_0, c_1 \geq 0$, although it was found earlier by Nakamura and Hirota [67] in connection with an auto-Bäcklund transformation for the mKdV equation. This equation has been studied by Guil Guerrero [35], who recovered it as a specialization of a system of zero-curvature equations, and Fokas [27], who found a recursion operator. It is known to be related to several famous equations for differing values of the parameters c_0 and c_1. If $c_0 = c_1 = 0$ then p and t' can be scaled to recover the PmKdV equation. When $c_0c_1 = 0$ with $(c_0, c_1) \neq (0, 0)$, this equation is related to equation (4.29) by another scaling of coordinates, so that its relationships with other equations, for these special situations, are described in Section 4.5. For the remaining case, where $c_0c_1 \neq 0$, one can make the change of coordinates

$$t = -2t'(c_0c_1)^{3/2}, \quad x = (x' + 6c_0c_1t')(c_0c_1)^{1/2}, \quad z = p + \tfrac{1}{2}\log(c_1/c_0),$$

which transforms the equation into

$$0 = z_t + z_{xxx} - \tfrac{1}{2}z_x^3 - 6z_x \cosh^2 z. \tag{5.34}$$

Equation (5.34) admits a potential q determined by

$$q_x = \sinh z, \quad q_t = -z_{xx}\cosh z + \tfrac{1}{2}z_x^2 \sinh z + 6\sinh z + 2\sinh^3 z,$$

and, although the resulting Wahlquist-Estabrook prolongation admits no nonlocal symmetry generators, it does feature the nonlocal partial symmetry generators $e^{2q}\partial_z$ and $e^{-2q}\partial_z$. By applying Proposition 5.8 with $\mathbf{u} = e^{2q}\partial_z$ one obtains the augmented prolongation described by the system of equations

$$\begin{aligned} r_x &= e^{2q}\cosh z, \\ r_t &= e^{2q}(-z_{xx}\sinh z - 2z_x + \tfrac{1}{2}z_x^2\cosh z + 6\cosh z + 2\cosh z\sinh^2 z). \end{aligned}$$

Not only does this prolongation admit the nonlocal symmetry generator

$$\mathbf{v}_1 = e^{2q}\partial_z + r\partial_q + \tfrac{1}{4}(e^{4q} + 4r^2)\partial_r,$$

but it also possesses a full internal symmetry group with generators $\mathbf{v}_2 = \partial_q + 2r\partial_r$ and $\mathbf{v}_3 = \partial_r$. Let G denote the symmetry group with infinitesimal generators $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$. It has Lie algebra isomorphic to $\mathfrak{sl}(2, \mathbb{R})$.

Solutions of the G-induced HC-projected problem associated with the augmented prolongation of equation (5.34) can be described by

$$z_x(x, t, z, q, r) = 2\cosh z + 2v(x, t)$$

for suitable functions $v(x,t)$. One finds that v must satisfy the mKdV equation

$$0 = v_t + v_{xxx} - 6v^2 v_x$$

so that equation (5.34) has the modified KdV equation as an M-projection. This construction almost exactly mimics that which recovers the KdV equation as an M-projection of the mKdV equation. Hence, equation (5.34) will be called the modified mKdV equation, abbreviated to the mmKdV equation. □

Section 3.3 ended with a description of an efficient method for constructing HC-projections. This section concludes with an analogous technique yielding M-projections. Suppose that G is a symmetry group, containing the internal symmetry group, of a Wahlquist-Estabrook prolongation of a system of differential equations with a full internal symmetry group. It follows from the full internal symmetry group property that all invariants of the prolonged G-action must be independent of the pseudopotentials. Thus the invariants can be functions of the jet variables only. If the original differential equation involves p independent variables, choose p invariants of the symmetry group action to act as independent variables for the M-projected system. Briefly, the idea is to construct the smallest possible number of additional invariants such that, after setting them equal to unknown functions of the p earlier invariants, all variables involved in the differential equation can be described in terms of $\dim G$ parametric variables (including the pseudopotentials) and derivatives of the unknown functions. The M-projected system arises from integrability conditions assumed when performing the calculations above and from the original differential equation. This process is demonstrated by the following example.

Example 5.20 The M-projection of Example 5.19 is rederived. Ignoring the coefficients of ∂_q and ∂_r, the prolongation of the nonlocal partial symmetry generator mentioned in that example is

$$\mathrm{pr}^{(n)}\mathbf{v} \;=\; e^{2q}\partial_z + 2e^{2q}\sinh z\,\partial_{z_x} + \cdots \;=\; e^{2q}\left(\partial_z + 2\sinh z\,\partial_{z_x} + \cdots\right).$$

The $p=2$ invariants to act as independent variables for the projected equation are chosen to be x and t and another invariant is easily seen to be $z_x - 2\cosh z$. Following the discussion above, let

$$z_x = 2\cosh z + 2v(x,t).$$

Using the chain rule to evaluate z_{xx} and z_{xxx}, and the mmKdV equation to solve for z_t, it follows that

$$z_t = 4(v^2+2)\cosh z - 4v_x \sinh z - 2v_{xx} + 4v^3 + 8v.$$

Finally, the integrability condition $z_{xt} - z_{tx} = 0$ requires that v satisfies the mKdV equation. □

The advantage of this method is that one does not need to augment the prolongation to find a true nonlocal symmetry generator equivalent to the partial one. Instead, prolong the nonlocal partial symmetry generator to $M^{(n)}$ for some n, and look for invariants as in Example 5.20. If there exists an M-projection involving a nonlocal symmetry generator equivalent to this nonlocal partial symmetry then the method described above should yield a new differential equation related to the original one by an M-projection.

5.5 Bäcklund transformations

In Section 5.4, M-projections of a differential equation Δ were constructed using nondegenerate Wahlquist-Estabrook prolongations (Δ, Ξ) of Δ which admit both a full internal symmetry group and a nonlocal symmetry generator. The case where (Δ, Ξ) possesses a nonlocal symmetry generator but not a full internal symmetry group is of independent interest and is treated here. Let G denote a subgroup of the symmetry group of (Δ, Ξ). The solutions of the differential equations Δ and $\Gamma = \Pi_G(\Delta, \Xi)$, the G-induced HC-projection of (Δ, Ξ), are related via the system (Δ, Ξ). With every solution of Δ one can associate a multi-parameter family of solutions to (Δ, Ξ) by solving the prolongation equations $\Xi[u, y] = 0$. This family projects onto a class of solutions to the new differential equation Γ. Clearly the process can be reversed. Starting with a solution to Γ there results a $(\dim G)$-parameter family of solutions to (Δ, Ξ) which in turn leads to a multi-parameter family of solutions of Δ. This relationship is an example of what has become known as a *Bäcklund transformation* and when the equations Δ and Γ coincide is called an *auto-Bäcklund transformation.* Motivation for this terminology is provided by Example 5.21.

There is considerable freedom in the choice of the group G used in constructing Γ. The approach adopted in this section will involve taking G to be the connected symmetry group of (Δ, Ξ) which has Lie algebra generated by a set containing all internal symmetry generators and at least one nonlocal generator of the prolonged system. The construction of Section 5.4 is thus considerably simplified. Given a prolongation admitting a nonlocal partial symmetry generator, one applies Proposition 5.8 repeatedly until, hopefully, a prolongation featuring a genuine nonlocal symmetry generator is obtained. The process stops there, whether or not the prolongation possesses a full internal symmetry group. One obtains the new differential equation by constructing an HC-projected equation of the augmented prolongation using the group G just described.

Example 5.21 The system of linear equations for $\{p(x,t), q(x,t)\}$,

$$\begin{aligned} \begin{pmatrix} p \\ q \end{pmatrix}_x &= \begin{pmatrix} 0 & 1 \\ -2u-\lambda & 0 \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix}, \\ \begin{pmatrix} p \\ q \end{pmatrix}_t &= \begin{pmatrix} 2u_x & -4u+4\lambda \\ 2(u_{xx}+4u^2)-4\lambda u-4\lambda^2 & -2u_x \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix}, \end{aligned} \tag{5.35}$$

is integrable if and only if $u(x,t)$ satisfies the KdV equation

$$0 = u_t + u_{xxx} + 12uu_x. \tag{5.36}$$

The Wahlquist-Estabrook prolongation of the KdV equation determined by equations (5.35) thus yields a zero-curvature representation. A straightforward calculation confirms that this prolongation admits exactly one nonlocal partial symmetry generator, $pq\partial_u$. This partial symmetry generator does not extend to a true symmetry generator of the prolonged system, so that one is forced to use the results of Proposition 5.8. This prolongation is augmented by introducing further pseudopotentials y and z described by

$$\begin{aligned} y_x &= z, \\ y_t &= 2u_x y - 4(u-\lambda)z - 2(2u+\lambda)p^2 - 2pq^2, \\ z_x &= -(2u+\lambda)y - 2p^2 q, \\ z_t &= (2u_{xx} + 4(u-\lambda)(2u+\lambda))y - 2u_x z \\ &\quad - 4u_x p^2 - 4(2u+\lambda)pq + 4(4u-\lambda)p^2 q - 2q^3. \end{aligned} \tag{5.37}$$

This prolongation contains some redundancy since, on solutions to equations (5.35) to (5.37), the function $p^4 - 2qy + 2pz$ is constant, indicating that a more efficient augmentation can be found. One such smaller prolongation is found by setting this constant to zero, so that $r = -2p^{-1}y$ satisfies

$$r_x = p^2, \quad r_t = 4(u + 2\lambda)p^2 + 4q^2. \tag{5.38}$$

The prolongation of the KdV equation described by equations (5.35) and (5.38) admits the genuine nonlocal symmetry generator

$$\mathbf{v}_1 = pq\partial_u - \tfrac{1}{2}pr\partial_p - \tfrac{1}{2}(p^3 + qr)\partial_q - \tfrac{1}{2}r^2\partial_r,$$

which is equivalent to $pq\partial_u$, together with the two internal symmetry generators

$$\mathbf{v}_2 = p\partial_p + q\partial_q + 2r\partial_r, \quad \mathbf{v}_3 = \partial_r.$$

The symmetry group with infinitesimal generators $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ has Lie algebra isomorphic to $\mathfrak{sl}(2, \mathbb{R})$ and yields solutions of the HC-projected problem associated with this prolongation described by

$$u(x, t, p, q, r) = \alpha(x, t) - p^{-2}q^2$$

for appropriate functions α. Substituting this *Ansatz* into the KdV equation yields the equation for α,

$$0 = \alpha_t + \alpha_{xxx} - 12\alpha\alpha_x - 12\lambda\alpha_x,$$

which implies that $\tilde{u}(x, t) = -\alpha(x, t) - \lambda$ is a solution of the KdV equation. Thus

$$u \mapsto \tilde{u} = -\alpha - \lambda = -u - p^{-2}q^2 - \lambda \tag{5.39}$$

is an auto-Bäcklund transformation for the KdV equation. This mapping corresponds to the well known auto-Bäcklund transformation. Introduce a new variable $w(x, t) = \int^x u(x', t)dx'$ which satisfies the potential KdV (PKdV) equation

$$0 = w_t + w_{xxx} + 6w_x^2.$$

Thus $u = w_x$ and equation (5.39) becomes

$$0 = \tilde{w}_x + w_x + p^{-2}q^2 + \lambda \tag{5.40}$$

which, using the prolongation equation defining q_x, reduces to

$$0 = \tilde{w}_x - w_x - p^{-1}q_x + p^{-2}q^2.$$

The equation defining p_x then allows one to write

$$0 = \tilde{w}_x - w_x - p^{-1}q_x + p^{-2}p_x q = (\tilde{w} - w - p^{-1}q)_x,$$

so that $p^{-1}q = \tilde{w} - w$, with the integration constant taken as zero. Equation (5.40) becomes

$$0 = (\tilde{w} + w)_x + (\tilde{w} - w)^2 + \lambda,$$

which is the familiar spatial member of the system of equations describing the auto-Bäcklund transformation for the PKdV equation [91]. □

Let G denote the connected symmetry group of (Δ, Ξ) with Lie algebra generated by a set containing all internal symmetry generators and at least one nonlocal symmetry generator of the prolonged system. Denote the internal symmetry group of (Δ, Ξ) by H. Since the action of H leaves all variables involved in Δ invariant and affects only pseudopotentials, the H-induced HC-projection of (Δ, Ξ) can be regarded as an intermediate Wahlquist-Estabrook prolongation of Δ, say (Δ, Λ). This yields the diagram

$$\begin{tikzcd}
 & & (\Delta,\Xi) \arrow[dl, "H"'] \arrow[dd, "G"] \\
 & (\Delta,\Lambda) \arrow[dl, dashed] \arrow[dr, dashed] & \\
\Delta & & \Gamma
\end{tikzcd}$$

which shows that Γ is a G/H-induced M-projected equation associated with (Δ, Λ). Thus, the differential equations Δ and Γ, regarded as being related by a Bäcklund transformation, have a common Wahlquist-Estabrook prolongation, (Δ, Λ).

It is illuminating to consider the auto-Bäcklund transformation of the KdV equation featured in Example 5.21 in light of this discussion.

Example 5.22 The prolongation of the KdV equation described above by equations (5.35) and (5.38) admits an internal symmetry group H generated by the vector fields $\mathbf{v}_2$ and $\mathbf{v}_3$ given in Example 5.21. A maximal, functionally independent set of invariants of the resulting group action on the space with coordinates (x, t, u, p, q, r) is $\{x, t, u, v = -p^{-1}q\}$, so that the H-induced HC-projected system comprises the KdV equation and the intermediate prolongation

$$\begin{aligned} v_x &= v^2 + 2u + \lambda, \\ v_t &= -4(u-\lambda)v^2 - 4u_x v - 2(u_{xx} + 4u^2 - 2\lambda u - 2\lambda^2). \end{aligned}$$

This prolongation is familiar from [92], which also included the auto-Bäcklund transformation of equation (5.39), written in the form

$$u \mapsto \tilde{u} = -u - v^2 - \lambda. \tag{5.41}$$

Wahlquist and Estabrook [92] have observed that the pseudopotential v must satisfy the equation

$$0 = v_t + v_{xxx} - 6v^2 v_x - 6\lambda v_x, \tag{5.42}$$

which is called the deformed mKdV (dmKdV) equation hereafter. It has been pointed out by Chen [10] that the auto-Bäcklund transformation, equation (5.41), can be decomposed into a discrete symmetry of the dmKdV equation, $v \mapsto \tilde{v} = -v$, followed by the original M-projection

$$u = \tfrac{1}{2}(v_x - v^2 - \lambda).$$

That is,

$$\tilde{u} = \tfrac{1}{2}(\tilde{v}_x - \tilde{v}^2 - \lambda) = -\tfrac{1}{2}v_x - \tfrac{1}{2}v^2 - \tfrac{1}{2}\lambda = -u - v^2 - \lambda.$$

At this point the auto-Bäcklund transformation can be represented by

$$\begin{tikzcd} \text{dmKdV} \arrow[r, "v \mapsto -v"] \arrow[d, "u=\frac{1}{2}(v_x - v^2 - \lambda)"'] & \text{dmKdV} \arrow[d, "u=\frac{1}{2}(v_x - v^2 - \lambda)"] \\ \text{KdV} & \text{KdV} \end{tikzcd}$$

where each vertical arrow corresponds to an application of the M-projection. This idea is the basis of Chen's construction of auto-Bäcklund transformations and will be referred to later in the current section. It is well known that the dmKdV equation is related to the mKdV equation by an invertible change of coordinates: if $v(x,t)$ is a solution to equation (5.42) then $v'(x,t) = v(x - 6\lambda t, t)$ satisfies

$$0 = v'_t + v'_{xxx} - 6(v')^2 v'_x.$$

Pirani, Robinson and Shadwick (Example 6.3 of [76]) have shown that the following diagram commutes:

$$\begin{tikzcd}
\mathrm{dmKdV} \arrow[r, "v(x,t)\mapsto v'(x,t)=v(x-6\lambda t,t)"] \arrow[d, "v(x,t)\mapsto u(x,t)=\frac{1}{2}(v_x-v^2-\lambda)"'] & \mathrm{mKdV} \arrow[d, "v'(x,t)\mapsto u'(x,t)=\frac{1}{2}(v'_x-(v')^2)"] \\
\mathrm{KdV} \arrow[r, "u(x,t)\mapsto u'(x,t)=u(x-6\lambda t,t)+\frac{1}{2}\lambda"] & \mathrm{KdV}
\end{tikzcd}$$

The bottom arrow corresponds to an application of the Galilean symmetry group $\{g_\lambda : \lambda \in \mathbb{R}\}$ of the KdV equation. Combining the results of Chen and Pirani *et al.*, the auto-Bäcklund transformation for the KdV equation can be decomposed into

$$\begin{tikzcd}
 & \mathrm{mKdV} \arrow[r, "v\mapsto -v"] \arrow[d, "v\mapsto u=\frac{1}{2}(v_x-v^2)"'] & \mathrm{mKdV} \arrow[d, "v\mapsto u=\frac{1}{2}(v_x-v^2)"] & \\
\mathrm{KdV} \arrow[r, "g_\lambda"] & \mathrm{KdV} & \mathrm{KdV} \arrow[r, "g_{-\lambda}"] & \mathrm{KdV}
\end{tikzcd}$$

where vertical arrows represent the standard Miura transformation. It is this decomposition which motivates the form used in Example 3.12 to describe an auto-Bäcklund transformation for the Harry Dym equation. □

The auto-Bäcklund transformation of the KdV equation was obtained in Example 5.21 following the discovery of a nonlocal partial symmetry generator of the prolongation of the KdV equation resulting from its zero-curvature formulation. Wahlquist and Estabrook constructed this auto-Bäcklund transformation using the intermediate prolongation of Example 5.22. That prolongation can admit no nonlocal partial symmetry generators because $pq\partial_u$ cannot be expressed as a vector field on a manifold with coordinates $(x, t, u, v = -p^{-1}q)$. Further, if this intermediate prolongation featured a nonlocal partial symmetry generator then, by the comments after Example 5.6, the overall prolongation would also admit that partial symmetry generator. It is important to remember this example since it shows the construction described in this section is not guaranteed to yield all Bäcklund transformations associated with a particular prolongation of a differential equation. In the case of the KdV equation, at least, this disadvantage is not as serious as one might suppose. The pseudopotential v is determined by the Riccati equation

$$v_x = v^2 + 2u + \lambda,$$

and similarly for the t-derivative of v. One can solve such equations by first linearizing them, making the change of variable $v = (-\log p)_x$ and, in so doing, recovering the prolongation of Example 5.21. Thus, in this case at least, one is naturally led to the prolongation for which the nonlocal symmetry approach is successful.

Although the search for Bäcklund transformations via nonlocal partial symmetry generators is not always successful, it is important to realize that it does have several major advantages over one traditional method for finding such transformations. Given a prolongation (Δ, Ξ) of the differential equation Δ, one can seek an auto-Bäcklund transformation of Δ by searching for mappings $\rho : M \times Y \to M$ which take solutions of (Δ, Ξ) onto solutions of Δ [92]. In terms of local coordinates, such a mapping will take the form

$$\tilde{x}^i = f^i(x, u, y), \quad \tilde{u}^\alpha = g^\alpha(x, u, y), \quad i = 1, \ldots, p, \quad \alpha = 1, \ldots, q, \tag{5.43}$$

and, given a solution $\{u(x), y(x)\}$ of (Δ, Ξ), the new solution of Δ will be given parametrically by

$$\tilde{x} = f(x, u(x), y(x)), \quad \tilde{u} = g(x, u(x), y(x)).$$

The differential equations determining f and g are highly nonlinear and notoriously difficult to solve. Contrast this with the determining equations for the partial symmetry generators of (Δ, Ξ) which are always linear and often straightforward to solve, as demonstrated in Example 5.6 and Appendix B.

One situation where this linearity is especially useful is in the determination of superposition principles associated with a particular auto-Bäcklund transformation of a differential equation. A superposition principle provides an algebraic means of describing repeated applications of an auto-Bäcklund transformation. In particular, one need only solve a system of differential equations analogous to equations (5.35) the first time the transformation is performed. Subsequent applications only involve simple algebraic expressions. Even in the case of the relatively simple KdV equation, Wahlquist and Estabrook were driven to comment that their approach involved "another tedious calculation (we certainly suspect there must be a neater way to obtain these results)" [92]. Contrast this with the construction of the superposition principle using nonlocal partial symmetry generators which is given in the next example. The calculations are still involved but, most importantly, all determining equations are now linear.

Example 5.23 An efficient means of constructing the superposition principle for the KdV equation involves the Wahlquist-Estabrook prolongation described by two copies of equations (5.35) featuring pseudopotentials p_a, q_a and corresponding parameters λ_a, where $a = 1, 2$. When $\lambda_1 \neq \lambda_2$ one finds two nonlocal partial symmetry generators, $p_1 q_1 \partial_u$ and $p_2 q_2 \partial_u$. The prolongation can be augmented, as in Example 5.21, by adding two new pseudopotentials $\{r_a : a = 1, 2\}$ described by the appropriately relabelled equations (5.38). Nonlocal symmetry generators of this extended prolongation are

$$\begin{aligned}
\mathbf{v}_1 &= p_1 q_1 \partial_u - \tfrac{1}{2} p_1 r_1 \partial_{p_1} - \tfrac{1}{2}(p_1^3 + q_1 r_1)\partial_{q_1} - \tfrac{1}{2} r_1^2 \partial_{r_1} - \frac{p_1(p_1 q_2 - p_2 q_1)}{2(\lambda_1 - \lambda_2)}\partial_{p_2} \\
&\quad - \frac{1}{2}\left(p_1^2 p_2 + \frac{q_1(p_1 q_2 - p_2 q_1)}{\lambda_1 - \lambda_2}\right)\partial_{q_2} - \frac{(p_1 q_2 - p_2 q_1)^2}{2(\lambda_1 - \lambda_2)^2}\partial_{r_2}, \\
\mathbf{w}_1 &= p_2 q_2 \partial_u - \frac{p_2(p_1 q_2 - p_2 q_1)}{2(\lambda_1 - \lambda_2)}\partial_{p_1} - \frac{1}{2}\left(p_2^2 p_1 + \frac{q_2(p_1 q_2 - p_2 q_1)}{\lambda_1 - \lambda_2}\right)\partial_{q_1} \\
&\quad - \frac{(p_1 q_2 - p_2 q_1)^2}{2(\lambda_1 - \lambda_2)^2}\partial_{r_1} - \tfrac{1}{2} p_2 r_2 \partial_{p_2} - \tfrac{1}{2}(p_2^3 + q_2 r_2)\partial_{q_2} - \tfrac{1}{2} r_2^2 \partial_{r_2},
\end{aligned}$$

and there also exist the following internal symmetry generators:

$$\begin{aligned}
\mathbf{v}_2 &= p_1 \partial_{p_1} + q_1 \partial_{q_1} + 2 r_1 \partial_{r_1}, \quad \mathbf{v}_3 = \partial_{r_1}, \\
\mathbf{w}_2 &= p_2 \partial_{p_2} + q_2 \partial_{q_2} + 2 r_2 \partial_{r_2}, \quad \mathbf{w}_3 = \partial_{r_2}.
\end{aligned}$$

The resulting Lie algebra is $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{R}) \oplus \mathfrak{sl}(2, \mathbb{R})$. Invariants on the space with coordinates $(x, t, u, p_1, q_1, r_1, p_2, q_2, r_2)$ are x, t and

$$\tilde{u} = u + \frac{(\lambda_1 - \lambda_2)(p_1 q_2 + p_2 q_1)}{p_1 q_2 - p_2 q_1} - \frac{(\lambda_1 - \lambda_2)^2 p_1^2 p_2^2}{(p_1 q_2 - p_2 q_1)^2}. \tag{5.44}$$

Whenever the KdV and all prolongation equations are satisfied, the function $\tilde{u}(x,t)$ described above is also a solution to the KdV equation. The mapping $u \mapsto \tilde{u}$ is the required superposition principle.

Suppose that the auto-Bäcklund transformation of Example 5.21, with $\lambda = \lambda_1$, is applied to the seed solution $u(x,t)$ and yields a solution $u_1(x,t)$. The auto-Bäcklund transformation with $\lambda = \lambda_2$ can then be applied to the seed solution $u_1(x,t)$, leading to another solution $u_{12}(x,t)$. Notice that when performing the second transformation, equations (5.35) are replaced by similar expressions with $u(x,t)$ replaced by $u_1(x,t)$ everywhere. This makes performing a sequence of auto-Bäcklund transformations very difficult, as new pseudopotentials must be calculated with each iteration. Equation (5.44) greatly simplifies the construction as one can show that $\tilde{u}(x,t) = u_{12}(x,t)$. That is, to apply a sequence of auto-Bäcklund transformations to $u(x,t)$ one can solve the two copies of equations (5.35) and substitute the resulting pseudopotentials into equation (5.44). Notice also that an identical solution $\tilde{u}(x,t)$ is obtained if the roles of λ_1 and λ_2 are reversed, so that the order in which the component transformations are applied is irrelevant. (See [91] and [92] for more details.) □

Another important advantage of the method advocated here is more subtle. In the traditional approach summarized before Example 5.23, it is necessary to make certain crucial assumptions about the form of the mapping ρ in order to reduce the complexity of the determining equations for f and g of equations (5.43). Frequently, assumptions such as $\tilde{x} = x$ are made but one can never be sure that such simplifications do not hide actual Bäcklund transformations. In any event, the choice of such an *Ansatz* is an *ad hoc* affair and, in some cases, Bäcklund transformations remained undiscovered since no appropriate special forms of ρ were tried. One example is that of the Harry Dym equation. The prolongation structure of this equation has been known for some time, but initial efforts by Leo, Leo, Soliani, Solombrino and Martina [61] to find a Bäcklund transformation using the traditional prolongation approach resulted in failure. The final example constructs an auto-Bäcklund transformation for the Harry Dym equation using nonlocal symmetries. As one will notice from the transformation, it is no surprise that the appropriate *Ansatz* was not tried by Leo *et al*!

Example 5.24 The differential equation due to Harry Dym

$$u_t = u^3 u_{xxx}$$

has the associated linear system

$$\begin{pmatrix} p \\ q \end{pmatrix}_x = \begin{pmatrix} 0 & 4\lambda \\ -\lambda u^{-2} & 0 \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix},$$

$$\begin{pmatrix} p \\ q \end{pmatrix}_t = \begin{pmatrix} 8\lambda^2 u_x & -64\lambda^3 u \\ 2\lambda u_{xx} + 16\lambda^3 u^{-1} & -8\lambda^2 u_x \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix},$$

which is gauge-equivalent to that used in [89] and [90] to solve the equation using the inverse scattering transform. This Wahlquist-Estabrook prolongation admits a single nonlocal partial symmetry generator

$$\mathbf{u} = p^2\partial_x + 8\lambda pqu\partial_u.$$

To augment the prolongation to one featuring a true nonlocal symmetry generator equivalent to $\mathbf{u}$ it proves sufficient to introduce one more pseudopotential r defined by the equations

$$r_x = q^2, \quad r_t = -4\lambda^2 u^{-1}p^2 + 4\lambda u_x pq - 32\lambda^2 uq^2.$$

The new prolongation admits the nonlocal symmetry generator

$$\mathbf{v}_1 = p^2\partial_x + 8\lambda pqu\partial_u + 16\lambda^2 pr\partial_p + 4\lambda q(4\lambda r - pq)\partial_q + 16\lambda^2 r^2\partial_r$$

and two internal symmetry generators, $\mathbf{v}_2 = p\partial_p + q\partial_q + 2r\partial_r$ and $\mathbf{v}_3 = \partial_r$. Once more, the vector fields $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ span a Lie algebra isomorphic to $\mathfrak{sl}(2,\mathbb{R})$. Invariants of the infinitesimal group action on $M \times Y$ are t, $4\lambda x - pq^{-1}$ and $p^{-2}q^2u$, so that one seeks solutions of the three-extended problem associated with the prolonged system described by

$$u(x,t,p,q,r) = p^2q^{-2} \cdot v(y,t),$$

where v is an undetermined function and $y = x - \frac{1}{4}\lambda^{-1}pq^{-1}$. Substituting this *Ansatz* into the Harry Dym equation indicates that v must be a solution of

$$0 = 64v^5v_t + v^2v_{yyy} - 6vv_yv_{yy} + 6v_y^3,$$

an equation which is equivalent to the Harry Dym equation. The invertible change of dependent variable $\tilde{u}(y,t) = -1/(4v(y,t))$ gives

$$\tilde{u}_t = \tilde{u}^3\tilde{u}_{yyy},$$

so that $u(x,t) \mapsto \tilde{u}(y,t)$ describes an auto-Bäcklund transformation for the Harry Dym equation. The author believes that this auto-Bäcklund transformation is new. In particular, it is different from the transformation featured in Example 3.12 and found earlier by Rogers and Wong [77]. □

Chen [10] has also derived auto-Bäcklund transformations from zero-curvature representations of differential equations. His method is different from that demonstrated here and relies on finding certain discrete symmetries of the differential equations satisfied by the pseudopotentials, as mentioned in Example 5.22. This suffers similar handicaps to the traditional construction of Bäcklund transformations, since the equations determining discrete symmetries of a system of differential equations are highly nonlinear. Once more one must make substantial assumptions about the form of these discrete symmetries before attempting to solve the determining equations. In practice, one does not even do this, but instead relies on the discrete symmetries being relatively trivial, such as a change in sign of one or more dependent variables.

The reader is referred to Appendix C where Bäcklund transformations are derived from nonlocal partial symmetry generators of zero-curvature representations of some famous differential equations. These results suggest that once a zero-curvature representation is known, the most expeditious route to constructing auto-Bäcklund transformations is via the calculation of nonlocal partial symmetry generators of these prolongations.

Chapter 6

Loop algebras and KdV equations

This chapter consists of an extended example examining the sophisticated algebraic structures which emerge when applying the inverse of the usual recursion operator for the KdV equation to the zero symmetry characteristic. One obtains a subalgebra of the loop algebra over $\mathfrak{sl}(2,\mathbb{R})$, a structure which occurs frequently in soliton theory (see [20], [25], [26], [92] and [97]). Section 6.1 constructs an infinite-dimensional Wahlquist-Estabrook prolongation of the KdV equation. The usual recursion operator for this equation is inverted in Section 6.2 and is used to construct three infinite families of nonlocal partial symmetry generators, which are then prolonged to genuine nonlocal symmetries. A symmetry algebra isomorphic to $\mathfrak{sl}(2,\mathbb{R})\otimes\mathbb{R}[\lambda,\lambda^{-1}]$ is found. Finally, Section 6.3 recovers similar results for the mKdV, PmKdV and PPmKdV equations. The role of zero-curvature representations and gauge transformations is also discussed there.

Most of the material presented in the current chapter also appears in [37]. The process of inverting the recursion operator and the derivation of the nonlocal symmetry generators are presented in more detail here while the part of [37] which deals with applications for the symmetries discovered appears elsewhere in the current work.

6.1 Infinite-dimensional prolongation of the KdV equation

The equations

$$\begin{aligned} y_x^a &= X_1^a(y) + 2uX_2^a(y), \\ y_t^a &= -2(u_{xx} + 4u^2)X_2^a(y) + 2u_x X_3^a(y) - 4uX_4^a(y) + 4X_5^a(y), \end{aligned} \tag{6.1}$$

are a special case of the system used by Wahlquist and Estabrook [92] to prolong the KdV equation

$$0 = u_t + u_{xxx} + 12uu_x.$$

Equations (6.1) define a Wahlquist-Estabrook prolongation if and only if the functions X_μ^a satisfy

$$[X_1, X_2] = X_3, \quad [X_1, X_3] = 2X_4, \quad [X_2, X_3] = -2X_2,$$
$$[X_1, X_4] = 2[X_2, X_5], \quad [X_1, X_5] = 0.$$

The resulting prolongation algebra $\mathcal{L}$ can be identified by suitably modifying the algebra determined by van Eck [21]. If $\mathfrak{g} \cong \mathfrak{sl}(2, \mathbb{R}) \otimes \mathbb{R}[\lambda]$ has basis $\{U_n, V_n, W_n : n = 0, 1, \ldots\}$ and commutator table

	U_n	V_n	W_n
U_m	0	$2V_{m+n}$	$-2W_{m+n}$
V_m		0	$-U_{m+n}$
W_m			0

and $\mathfrak{i}$ is the Abelian Lie algebra with basis $\{Z_1, Z_2\}$ then the mapping

$$\begin{aligned} \mathcal{L} &\to \mathfrak{g} \oplus \mathfrak{i}, \\ X_1 &\mapsto V_1 + W_0 + Z_1, \\ X_2 &\mapsto V_0, \\ X_3 &\mapsto U_0, \\ X_4 &\mapsto -V_1 + W_0, \\ X_5 &\mapsto V_2 + W_1 + Z_2, \end{aligned}$$

is an isomorphism. A sequence of Wahlquist-Estabrook prolongations of the KdV equation will be constructed from various representations of $\mathfrak{g}$ using Proposition 4.5. Recall that the standard faithful representation of $\mathfrak{g}$ is as the space of matrices in $\mathfrak{sl}(2,\mathbb{R})$ with components which are Taylor series in some parameter λ. This representation is given by

$$U_m \mapsto \begin{pmatrix} \lambda^m & 0 \\ 0 & -\lambda^m \end{pmatrix}, \quad V_m \mapsto \begin{pmatrix} 0 & -\lambda^m \\ 0 & 0 \end{pmatrix}, \quad W_m \mapsto \begin{pmatrix} 0 & 0 \\ \lambda^m & 0 \end{pmatrix}, \tag{6.2}$$

where the Lie bracket on the representation space is $[\mathbf{M},\mathbf{N}] = \mathbf{MN} - \mathbf{NM}$.

For each nonnegative integer N let $G^{(N)}$ denote the submanifold of $\mathbb{R}^{4(N+1)}$ comprising those points $(p_n, q_n, r_n, s_n : n = 0, \ldots, N)$ such that

$$\sum_{i=0}^{n}(p_i s_{n-i} - q_i r_{n-i}) = \delta_0^n, \quad n = 0, \ldots, N. \tag{6.3}$$

The binary operation

$$G^{(N)} \times G^{(N)} \to G^{(N)}, \quad ((p_n, q_n, r_n, s_n), (\bar{p}_n, \bar{q}_n, \bar{r}_n, \bar{s}_n)) \mapsto (p'_n, q'_n, r'_n, s'_n),$$

defined by

$$\begin{aligned} p'_n &= \sum_{i=0}^{n}(p_i \bar{p}_{n-i} + q_i \bar{r}_{n-i}), \\ q'_n &= \sum_{i=0}^{n}(p_i \bar{q}_{n-i} + q_i \bar{s}_{n-i}), \\ r'_n &= \sum_{i=0}^{n}(r_i \bar{p}_{n-i} + s_i \bar{r}_{n-i}), \\ s'_n &= \sum_{i=0}^{n}(r_i \bar{q}_{n-i} + s_i \bar{s}_{n-i}), \end{aligned}$$

gives $G^{(N)}$ the structure of a $3(N+1)$-dimensional Lie group. The multiplicative identity is the point with $p_0 = s_0 = 1$ and all other components zero and, at this point, the tangent space to $G^{(N)}$ is spanned by the vector fields $\{\partial_{p_i} - \partial_{s_i}, \partial_{q_i}, \partial_{r_i} : i = 0, \ldots, N\}$. The space of left-invariant vector fields on $G^{(N)}$, denoted $\mathfrak{g}_L^{(N)}$, has basis $\{\mathbf{u}_n^{*(N)}, \mathbf{v}_n^{*(N)}, \mathbf{w}_n^{*(N)} : n = 0, \ldots, N\}$ where

$$\mathbf{u}_n^{*(N)} = \sum_{i=0}^{N-n}(p_i \partial_{p_{i+n}} - q_i \partial_{q_{i+n}} + r_i \partial_{r_{i+n}} - s_i \partial_{s_{i+n}}),$$

$$\begin{aligned}
\mathbf{v}_n^{*(N)} &= -\sum_{i=0}^{N-n}(p_i\partial_{q_{i+n}} + r_i\partial_{s_{i+n}}),\\
\mathbf{w}_n^{*(N)} &= \sum_{i=0}^{N-n}(q_i\partial_{p_{i+n}} + s_i\partial_{r_{i+n}}).
\end{aligned}$$

Similarly, the space of right-invariant vector fields on $G^{(N)}$, denoted $\mathfrak{g}_R^{(N)}$, has basis $\{\mathbf{u}_n^{(N)}, \mathbf{v}_n^{(N)}, \mathbf{w}_n^{(N)} : n = 0, \ldots, N\}$ with

$$\begin{aligned}
\mathbf{u}_n^{(N)} &= \sum_{i=0}^{N-n}(p_i\partial_{p_{i+n}} + q_i\partial_{q_{i+n}} - r_i\partial_{r_{i+n}} - s_i\partial_{s_{i+n}}),\\
\mathbf{v}_n^{(N)} &= -\sum_{i=0}^{N-n}(r_i\partial_{p_{i+n}} + s_i\partial_{q_{i+n}}),\\
\mathbf{w}_n^{(N)} &= \sum_{i=0}^{N-n}(p_i\partial_{r_{i+n}} + q_i\partial_{s_{i+n}}).
\end{aligned}$$

For each nonnegative integer N the mapping

$$\begin{aligned}
\mathfrak{g} &\to \mathfrak{g}_L^{(N)},\\
U_m &\mapsto \begin{cases} \mathbf{u}_m^{*(N)}, & 0 \le m \le N,\\ 0, & m > N,\end{cases}\\
V_m &\mapsto \begin{cases} \mathbf{v}_m^{*(N)}, & 0 \le m \le N,\\ 0, & m > N,\end{cases}\\
W_m &\mapsto \begin{cases} \mathbf{w}_m^{*(N)}, & 0 \le m \le N,\\ 0, & m > N,\end{cases}
\end{aligned}$$

is a representation of $\mathfrak{g}$ which, by Proposition 4.5, determines a Wahlquist-Estabrook prolongation of the KdV equation which admits a full internal symmetry group. This prolongation is described by the prolongation equations

$$\begin{aligned}
p_{n,x} &= q_n,\\
q_{n,x} &= -2up_n - (1-\delta_n^0)p_{n-1},\\
r_{n,x} &= s_n,\\
s_{n,x} &= -2ur_n - (1-\delta_n^0)r_{n-1},\\
p_{n,t} &= 2u_xp_n - 4uq_n + 4(1-\delta_n^0)q_{n-1},\\
q_{n,t} &= 2(u_{xx}+4u^2)p_n - 2u_xq_n - 4(1-\delta_n^0)up_{n-1} - 4(1-\delta_n^0)(1-\delta_n^1)p_{n-2},\\
r_{n,t} &= 2u_xr_n - 4us_n + 4(1-\delta_n^0)s_{n-1},\\
s_{n,t} &= 2(u_{xx}+4u^2)r_n - 2u_xs_n - 4(1-\delta_n^0)ur_{n-1} - 4(1-\delta_n^0)(1-\delta_n^1)r_{n-2},
\end{aligned} \tag{6.4}$$

for all $n \in \{0, \ldots, N\}$, together with equations (6.3). The latter algebraic constraints arise from the embedding of $G^{(N)}$ in $\mathbb{R}^{4(N+1)}$. They also reflect the fact that the prolongation described by equations (6.4), alone, includes several redundant pseudopotentials, since

$$\tilde{D}_x\left(\sum_{i=0}^{N}(p_i s_{n-i} - q_i r_{n-i})\right) = \tilde{D}_t\left(\sum_{i=0}^{N}(p_i s_{n-i} - q_i r_{n-i})\right) = 0, \quad n = 0, \ldots, N.$$

By Proposition 4.5, the internal symmetry algebra of the prolongation described by equations (6.3) and (6.4) contains $\mathfrak{g}_R^{(N)}$ as a subalgebra.

A useful interpretation of $G^{(N)}$ results from considering the set of 2×2 matrices with components which are Taylor series in some parameter λ. A typical element is

$$g = \begin{pmatrix} \sum_{n=0}^{\infty} \lambda^n p_n & \sum_{n=0}^{\infty} \lambda^n q_n \\ \sum_{n=0}^{\infty} \lambda^n r_n & \sum_{n=0}^{\infty} \lambda^n s_n \end{pmatrix} \tag{6.5}$$

for real numbers $\{p_n, q_n, r_n, s_n : n = 0, 1, \ldots\}$ and, by treating the components of g as formal power series,

$$\begin{aligned} \det g &= \left(\sum_{n=0}^{\infty} \lambda^n p_n\right)\left(\sum_{n=0}^{\infty} \lambda^n s_n\right) - \left(\sum_{n=0}^{\infty} \lambda^n q_n\right)\left(\sum_{n=0}^{\infty} \lambda^n r_n\right) \\ &= \sum_{n=0}^{\infty} \lambda^n \sum_{i=0}^{n}(p_i s_{n-i} - q_i r_{n-i}). \end{aligned}$$

For each nonnegative integer N an equivalence relation can be defined on this set by regarding two matrices of the form given by equation (6.5) as equivalent if and only if they coincide to order N in λ. The point $(p_n, q_n, r_n, s_n : n = 0, \ldots, N) \in G^{(N)}$ can then be identified with the equivalence class containing the matrix

$$\begin{pmatrix} \sum_{n=0}^{N} \lambda^n p_n & \sum_{n=0}^{N} \lambda^n q_n \\ \sum_{n=0}^{N} \lambda^n r_n & \sum_{n=0}^{N} \lambda^n s_n \end{pmatrix}.$$

Equations (6.3) show that $G^{(N)}$ is thus identified with the collection of equivalence classes containing matrices g such that $\det g = 1 + \mathrm{O}(\lambda^{N+1})$. Notice that the multiplication operation defined on $G^{(N)}$ corresponds to ordinary matrix multiplication.

Also, by evaluating $\{\mathbf{u}_n^{*(N)}, \mathbf{v}_n^{*(N)}, \mathbf{w}_n^{*(N)} : n = 0, \ldots, N\}$ at the identity element in $G^{(N)}$, the identification above extends to

$$\mathbf{u}_m^{*(N)} \mapsto \begin{pmatrix} \lambda^m & 0 \\ 0 & -\lambda^m \end{pmatrix}, \quad \mathbf{v}_m^{*(N)} \mapsto \begin{pmatrix} 0 & -\lambda^m \\ 0 & 0 \end{pmatrix}, \quad \mathbf{w}_m^{*(N)} \mapsto \begin{pmatrix} 0 & 0 \\ \lambda^m & 0 \end{pmatrix},$$

for all $m \in \{0, \ldots, N\}$.

The prolongation of the KdV equation which is used in Sections 6.2 and 6.3 is obtained by taking the limit as $N \to \infty$ of the $G^{(N)}$-induced Wahlquist-Estabrook prolongation described above. It involves pseudopotentials $\{p_n, q_n, r_n, s_n : n = 0, 1, \ldots\}$ which are determined by equations (6.4) and the algebraic constraints $\{\Delta^n = 0 : n = 0, 1, \ldots\}$, where

$$\Delta^n = \delta_0^n - \sum_{i=0}^{n} (p_i s_{n-i} - q_i r_{n-i}), \quad n = 0, 1, \ldots .$$

The algebras of left- and right-invariant vector fields on $G^{(N)}$ lead to the algebras $\mathfrak{g}_L = \lim_{N\to\infty} \mathfrak{g}_L^{(N)}$ and $\mathfrak{g}_R = \lim_{N\to\infty} \mathfrak{g}_R^{(N)}$ of vector fields on the infinite-dimensional pseudopotential space. The observation that the natural projection $G^{(N+1)} \to G^{(N)}$ induces homomorphisms $\mathfrak{g}_L^{(N+1)} \to \mathfrak{g}_L^{(N)}$ and $\mathfrak{g}_R^{(N+1)} \to \mathfrak{g}_R^{(N)}$ is necessary for these limits to make sense. $\mathfrak{g}_L$ has basis $\{\mathbf{u}_n^*, \mathbf{v}_n^*, \mathbf{w}_n^* : n = 0, 1, \ldots\}$, where

$$\begin{aligned}
\mathbf{u}_n^* &= \sum_{i=0}^{\infty} (p_i \partial_{p_{i+n}} - q_i \partial_{q_{i+n}} + r_i \partial_{r_{i+n}} - s_i \partial_{s_{i+n}}), \\
\mathbf{v}_n^* &= -\sum_{i=0}^{\infty} (p_i \partial_{q_{i+n}} + r_i \partial_{s_{i+n}}), \\
\mathbf{w}_n^* &= \sum_{i=0}^{\infty} (q_i \partial_{p_{i+n}} + s_i \partial_{r_{i+n}}),
\end{aligned}$$

and commutator table

	$\mathbf{u}_n^*$	$\mathbf{v}_n^*$	$\mathbf{w}_n^*$
$\mathbf{u}_m^*$	0	$2\mathbf{v}_{m+n}^*$	$-2\mathbf{w}_{m+n}^*$
$\mathbf{v}_m^*$		0	$-\mathbf{u}_{m+n}^*$
$\mathbf{w}_m^*$			0

while $\mathfrak{g}_R$ has basis $\{\mathbf{u}_n, \mathbf{v}_n, \mathbf{w}_n : n = 0, 1, \ldots\}$, with

$$\mathbf{u}_n = \sum_{i=0}^{\infty} (p_i \partial_{p_{i+n}} + q_i \partial_{q_{i+n}} - r_i \partial_{r_{i+n}} - s_i \partial_{s_{i+n}}),$$

$$\begin{aligned}\mathbf{v}_n &= -\sum_{i=0}^{\infty}(r_i\partial_{p_{i+n}} + s_i\partial_{q_{i+n}}),\\ \mathbf{w}_n &= \sum_{i=0}^{\infty}(p_i\partial_{r_{i+n}} + q_i\partial_{s_{i+n}}),\end{aligned}$$

and commutator table

	$\mathbf{u}_n$	$\mathbf{v}_n$	$\mathbf{w}_n$
$\mathbf{u}_m$	0	$-2\mathbf{v}_{m+n}$	$2\mathbf{w}_{m+n}$
$\mathbf{v}_m$		0	$\mathbf{u}_{m+n}$
$\mathbf{w}_m$			0

Recall that the representation $\mathfrak{g} \to \mathfrak{g}_L^{(N)}$ combines with the results of Proposition 4.5 to yield the finite-dimensional prolongation with pseudopotential space $G^{(N)}$. This prolongation has full internal symmetry group and internal symmetry algebra with subalgebra $\mathfrak{g}_R^{(N)}$. These results are preserved by the limiting process insofar as equations (6.4) result from the representation

$$\mathfrak{g} \to \mathfrak{g}_L, \quad U_m \mapsto \mathbf{u}_m^*, \quad V_m \mapsto \mathbf{v}_m^*, \quad W_m \mapsto \mathbf{w}_m^*, \quad m = 0, 1, \ldots\ ,$$

and every vector in $\mathfrak{g}_R$ is an internal symmetry generator of the prolongation determined by equations (6.4) and $\{\Delta^n = 0 : n = 0, 1, \ldots\}$. The latter result is easily confirmed by direct calculation. For the remainder of this chapter $\mathfrak{g}_R$ will be denoted by $\mathcal{L}^{\mathrm{KdV}}_{\mathrm{Internal}}$.

There is another derivation of the infinite-dimensional prolongation being considered in this chapter. It is included because when one knows a zero-curvature representation of the differential equation being studied this method is much easier to implement than the one above. With the new method the internal symmetry generators need to be explicitly calculated, as the results of Proposition 4.5 are no longer relevant. Once more, the ubiquitous Taylor series in λ plays an important role.

The standard representation of $\mathfrak{sl}(2,\mathbb{R}) \otimes \mathbb{R}[\lambda]$, given by equation (6.2), yields a zero-curvature representation of the KdV equation. Since the Lie bracket in the representation space is $[\mathbf{M},\mathbf{N}] = \mathbf{MN} - \mathbf{NM}$, the associated linear system is

$$\mathbf{y}_x^T = \mathbf{y}^T\mathbf{A}, \quad \mathbf{y}_t^T = \mathbf{y}^T\mathbf{B},$$

where

$$\begin{aligned} \mathbf{A} &= \begin{pmatrix} 0 & -2u \\ 1 & 0 \end{pmatrix} + \lambda \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}, \\ \mathbf{B} &= \begin{pmatrix} 2u_x & 2u_{xx} + 8u^2 \\ -4u & -2u_x \end{pmatrix} + \lambda \begin{pmatrix} 0 & -4u \\ 4 & 0 \end{pmatrix} + \lambda^2 \begin{pmatrix} 0 & -4 \\ 0 & 0 \end{pmatrix}. \end{aligned} \tag{6.6}$$

Taking transposes of these equations recovers the zero-curvature representation used in Section 5.5 to calculate an auto-Bäcklund transformation for the KdV equation. The extension of $\mathbf{y}$ to a matrix-valued function is certainly permissible — replacing $\mathbf{y}$ by $\mathbf{\Psi}^T$, say. This leads to the linear system

$$\mathbf{\Psi}_x = \mathbf{\Psi A}, \quad \mathbf{\Psi}_t = \mathbf{\Psi B},$$

with integrability condition

$$\mathbf{0} = D_t(\mathbf{A}) - D_x(\mathbf{B}) - \mathbf{AB} + \mathbf{BA}$$

which is equivalent to the KdV equation. $\mathbf{\Psi}$ will be treated as a function taking values in the Lie group $SL(2,\mathbb{R})$ and is expanded in a Taylor series about $\lambda = 0$. Thus

$$\mathbf{\Psi} = \sum_{n=0}^{\infty} \lambda^n \mathbf{\Psi}_n,$$

so that the prolongation equations become

$$\begin{aligned} \sum_{n=0}^{\infty} \lambda^n (\mathbf{\Psi}_n)_x = \mathbf{\Psi}_x = \mathbf{\Psi A} &= \left(\sum_{n=0}^{\infty} \lambda^n \mathbf{\Psi}_n \right) (\mathbf{A}_0 + \lambda \mathbf{A}_1) \\ &= \mathbf{\Psi}_0 \mathbf{A}_0 + \sum_{n=1}^{\infty} \lambda^n \left(\mathbf{\Psi}_n \mathbf{A}_0 + \mathbf{\Psi}_{n-1} \mathbf{A}_1 \right) \end{aligned}$$

and

$$\begin{aligned} \sum_{n=0}^{\infty} \lambda^n (\mathbf{\Psi}_n)_t = \mathbf{\Psi}_t = \mathbf{\Psi B} &= \left(\sum_{n=0}^{\infty} \lambda^n \mathbf{\Psi}_n \right) \left(\mathbf{B}_0 + \lambda \mathbf{B}_1 + \lambda^2 \mathbf{B}_2 \right) \\ &= \mathbf{\Psi}_0 \mathbf{B}_0 + \lambda (\mathbf{\Psi}_1 \mathbf{B}_0 + \mathbf{\Psi}_0 \mathbf{B}_1) \\ &\quad + \sum_{n=2}^{\infty} \lambda^n \left(\mathbf{\Psi}_n \mathbf{B}_0 + \mathbf{\Psi}_{n-1} \mathbf{B}_1 + \mathbf{\Psi}_{n-2} \mathbf{B}_2 \right), \end{aligned}$$

where $\mathbf{A} = \mathbf{A}_0 + \lambda \mathbf{A}_1$ and $\mathbf{B} = \mathbf{B}_0 + \lambda \mathbf{B}_1 + \lambda^2 \mathbf{B}_2$ are the expansions of $\mathbf{A}$ and $\mathbf{B}$ as polynomials in λ. Equating coefficients of the various powers of λ in these equations yields the system of linear equations

$$\begin{aligned}
(\boldsymbol{\Psi}_n)_x &= \boldsymbol{\Psi}_n \mathbf{A}_0 + (1-\delta_n^0)\boldsymbol{\Psi}_{n-1}\mathbf{A}_1, \\
(\boldsymbol{\Psi}_n)_t &= \boldsymbol{\Psi}_n \mathbf{B}_0 + (1-\delta_n^0)\boldsymbol{\Psi}_{n-1}\mathbf{B}_1 + (1-\delta_n^0)(1-\delta_n^1)\boldsymbol{\Psi}_{n-2}\mathbf{B}_2,
\end{aligned}$$

for all $n = 0, 1, \ldots$. Upon making the explicit parametrization

$$\boldsymbol{\Psi}_n = \begin{pmatrix} p_n & q_n \\ r_n & s_n \end{pmatrix}, \quad n = 0, 1, \ldots,$$

this system of equations reduces to the first order equations presented earlier. The algebraic constraints $\Delta^n = 0$ amount to the condition

$$\begin{aligned}
1 = \det \boldsymbol{\Psi} &= \left(\sum_{n=0}^{\infty} \lambda^n p_n\right)\left(\sum_{n=0}^{\infty} \lambda^n s_n\right) - \left(\sum_{n=0}^{\infty} \lambda^n q_n\right)\left(\sum_{n=0}^{\infty} \lambda^n r_n\right) \\
&= \sum_{n=0}^{\infty} \lambda^n \sum_{i=0}^{n} (p_i s_{n-i} - q_i r_{n-i}),
\end{aligned}$$

which must hold for all values of λ. That is,

$$\sum_{i=0}^{n} (p_i s_{n-i} - q_i r_{n-i}) = \delta_0^n, \quad n = 0, 1, \ldots .$$

This construction can be repeated for any differential equation admitting a zero-curvature representation involving a parameter λ. All that will change are the Lie algebra and group in which the appropriate matrices take their values. Section 6.3 considers the relationships between zero-curvature representations for various equations related to the KdV equation. This method of prolonging a differential equation proves very useful when constructing the nonlocal symmetries of that equation using the inverse of a recursion operator, as will be demonstrated in subsequent sections.

6.2 Nonlocal symmetries of the KdV equation

An infinite-dimensional non-Abelian algebra of nonlocal symmetry generators of the KdV equation which is isomorphic to a subalgebra of $\mathfrak{sl}(2,\mathbb{R}) \otimes \mathbb{R}[\lambda, \lambda^{-1}]$ is presented in this section. Calculations involved in this construction split into two categories. The first type involves informal calculations which suggest the form which the nonlocal symmetry generators will take. These motivating calculations are presented in some detail so that, hopefully, the reader can derive similar families of symmetry generators for other equations of interest. The second type of calculation arises when formally proving that these generators are true nonlocal symmetries and mainly involves tedious algebraic manipulation. Such algebra is readily duplicated and most details will be found in Appendix D.

Olver [71] proved that the integro-differential operator

$$\mathcal{R} = D_x^2 + 8u + 4u_x D_x^{-1}$$

satisfies the traditional requirements of a recursion operator for the KdV equation. That is, if $Q \in \mathfrak{A}$ is the characteristic of a generalized symmetry $\mathbf{v}_Q$ of the KdV equation, so that

$$(D_t + D_x^3 + 12uD_x + 12u_x)(Q) = 0$$

on solutions to that equation, then $\mathbf{v}_{\mathcal{R}(Q)}$ is also a generalized symmetry. In terms of Section 5.3, if $Q \in \tilde{\mathfrak{A}}$ is the characteristic of a generalized partial symmetry generator of a prolongation of the KdV equation to $M \times Y$, then

$$(\tilde{D}_t + \tilde{D}_x^3 + 12u\tilde{D}_x + 12u_x)(Q) = 0 \tag{6.7}$$

on $\mathcal{S}_\Delta \times Y$, so that the system

$$\tilde{D}_x(P) = Q, \quad \tilde{D}_t(P) = (-\tilde{D}_x^2 + 12u)(Q),$$

for $P \in \tilde{\mathfrak{A}}$ is integrable. Then

$$\mathcal{R}(Q) = (\tilde{D}_x^3 + 8u\tilde{D}_x + 4u_x)(P)$$

is the new symmetry characteristic.

This recursion operator can be inverted using the results of Fordy and Gibbons [30]. They proved that $\mathcal{R}$ admits the factorization

$$\mathcal{R} = (\tilde{D}_x - 2v)\tilde{D}_x(\tilde{D}_x + 2v)\tilde{D}_x^{-1}$$

where the pseudopotential v is defined by

$$v_x = v^2 + 2u, \quad v_t = -4uv^2 - 4u_x v - 2(u_{xx} + 4u^2). \tag{6.8}$$

The usual way to linearize these Riccati equations involves the substitution $v = -p_0^{-1}q_0$ where p_0 and q_0 are two of the pseudopotentials introduced in Section 6.1. Thus

$$\mathcal{R} = (\tilde{D}_x + 2p_0^{-1}q_0)\tilde{D}_x(\tilde{D}_x - 2p_0^{-1}q_0)\tilde{D}_x^{-1},$$

an expression which can be simplified further. Since

$$(\tilde{D}_x + 2p_0^{-1}q_0)\cdot p_0^{-2} = p_0^{-2}\tilde{D}_x, \quad (\tilde{D}_x - 2p_0^{-1}q_0)\cdot p_0^2 = p_0^2\tilde{D}_x,$$

it follows that

$$\tilde{D}_x + 2p_0^{-1}q_0 = p_0^{-2}\tilde{D}_x p_0^2, \quad \tilde{D}_x - 2p_0^{-1}q_0 = p_0^2\tilde{D}_x p_0^{-2}.$$

Thus

$$\mathcal{R} = p_0^{-2}\tilde{D}_x p_0^2\tilde{D}_x p_0^2\tilde{D}_x p_0^{-2}\tilde{D}_x^{-1}$$

and

$$\mathcal{R}^{-1} = \tilde{D}_x p_0^2\tilde{D}_x^{-1}p_0^{-2}\tilde{D}_x^{-1}p_0^{-2}\tilde{D}_x^{-1}p_0^2.$$

In order to express $\mathcal{R}^{-1}$ in the form required by Definition 5.15, one must first solve the system

$$\tilde{D}_x(F) = p_0^2 Q, \quad \tilde{D}_t(F) = (-p_0^2\tilde{D}_x^2 + 2p_0q_0\tilde{D}_x - 2(4up_0^2 + q_0^2))(Q), \tag{6.9}$$

for $F \in \tilde{\mathfrak{A}}$ which is integrable due to Q satisfying equation (6.7) whenever $\mathbf{v}_Q$ is a generalized partial symmetry generator. Next, $G \in \tilde{\mathfrak{A}}$ must satisfy

$$\tilde{D}_x(G) = p_0^{-2}F, \quad \tilde{D}_t(G) = (-\tilde{D}_x + 2p_0^{-1}q_0)(Q) - 4up_0^{-2}F, \tag{6.10}$$

and $H \in \tilde{\mathfrak{A}}$ must be a solution to

$$\tilde{D}_x(H) = p_0^{-2}G, \quad \tilde{D}_t(H) = -p_0^{-2}Q - 4up_0^{-2}G. \tag{6.11}$$

The new characteristic is thus

$$\begin{aligned}
\mathcal{R}^{-1}(Q) &= \tilde{D}_x p_0^2\tilde{D}_x^{-1}p_0^{-2}\tilde{D}_x^{-1}(p_0^{-2}F) \\
&= \tilde{D}_x p_0^2\tilde{D}_x^{-1}(p_0^{-2}G) \\
&= \tilde{D}_x(p_0^2 H) \\
\mathcal{R}^{-1}(Q) &= G + 2p_0q_0H.
\end{aligned}$$

A simple calculation shows that whenever equations (6.7) and (6.9) to (6.11) are satisfied

$$(\tilde{D}_t + \tilde{D}_x^3 + 12u\tilde{D}_x + 12u_x)(G + 2p_0q_0H) = 0$$

on $\mathcal{S}_\Delta \times Y$. Thus $\mathcal{R}^{-1}(Q)$ is the characteristic of a generalized partial symmetry generator of the KdV equation.

The trivial symmetry of the KdV equation has characteristic equal to zero and, applying $\mathcal{R}^{-1}$ to this characteristic, one obtains equations for F

$$\tilde{D}_x(F) = \tilde{D}_t(F) = 0,$$

which force $F = c_1$, with c_1 an arbitrary constant. G must now satisfy

$$\tilde{D}_x(G) = c_1p_0^{-2}, \quad \tilde{D}_t(G) = -4c_1up_0^{-2},$$

and, since

$$\begin{aligned} \tilde{D}_x(p_0^{-1}r_0) &= p_0^{-2}(1 - \Delta^0) &&= p_0^{-2}, \\ \tilde{D}_t(p_0^{-1}r_0) &= -4up_0^{-2}(1 - \Delta^0) &&= -4up_0^{-2}, \end{aligned}$$

on solutions to the algebraic constraints $\{\Delta^n = 0 : n = 0, 1, \ldots\}$, it follows that $G = c_1p_0^{-1}r_0 + c_2$ for an arbitrary constant c_2. The equations determining H are

$$\begin{aligned} \tilde{D}_x(H) &= c_1p_0^{-3}r_0 + c_2p_0^{-2} &&= c_1(p_0^{-2})(p_0^{-1}r_0) + c_2p_0^{-2}, \\ \tilde{D}_t(H) &= -4c_1up_0^{-3}r_0 - 4c_2up_0^{-2} &&= c_1(-4up_0^{-2})(p_0^{-1}r_0) - 4c_2up_0^{-2}, \end{aligned}$$

so that $H = \frac{1}{2}c_1(p_0^{-1}r_0)^2 + c_2p_0^{-1}r_0 + c_3$, with c_3 another arbitrary constant. Thus

$$\mathcal{R}^{-1}(0) = c_1p_0^{-1}r_0(p_0s_0 + \Delta^0) + c_2(p_0s_0 + q_0r_0 + \Delta^0) + 2c_3p_0q_0$$

and, since $\Delta^0 = 0$ on solutions to the prolongation equations, it follows that the new nonlocal partial symmetry generator of the KdV equation is

$$(c_1(r_0s_0) + c_2(p_0s_0 + q_0r_0) + 2c_3(p_0q_0))\partial_u,$$

yielding three new nonlocal partial symmetries. Notice that they are "geometric" rather than "generalized" vector fields.

This process can be continued indefinitely by applying $\mathcal{R}^{-1}$ repeatedly to the characteristics r_0s_0, $p_0s_0 + q_0r_0$ and p_0q_0 in turn. With $Q = p_0q_0$ it is easily shown that

$$F = \tfrac{1}{4}p_0^4, \quad G = \tfrac{1}{4}(p_1q_0 - p_0q_1), \quad H = -\tfrac{1}{4}p_0^{-1}p_1,$$

and so

$$\mathcal{R}^{-1}(p_0q_0) = -\tfrac{1}{4}(p_0q_1 + p_1q_0).$$

The only step in the calculations which may not be transparent is the observation that

$$\tilde{D}_x(p_1q_0 - p_0q_1) = p_0^2. \tag{6.12}$$

Also, integration constants have not been introduced when evaluating F, G and H since all they will yield are reappearances of the three characteristics found earlier.

After this and perhaps several more calculations a pattern emerges from the characteristics resulting from recursive application of $\mathcal{R}^{-1}$ to p_0q_0. The results are embodied in the following theorem.

Theorem 6.1 *For all $n = 1, 2, \ldots$, the vector fields $\xi^{(n)}\partial_u$, $\phi^{(n)}\partial_u$ and $\theta^{(n)}\partial_u$ with*

$$\begin{aligned} \xi^{(n)} &= 2\sum_{i=0}^{n-1}(p_is_{n-1-i} + q_ir_{n-1-i}), \\ \phi^{(n)} &= -2\sum_{i=0}^{n-1} r_is_{n-1-i}, \\ \theta^{(n)} &= -2\sum_{i=0}^{n-1} p_iq_{n-1-i}, \end{aligned}$$

are nonlocal partial symmetry generators of the KdV equation. The characteristics are related by

$$\begin{aligned} \mathcal{R} : \xi^{(n)} &\mapsto -4(1-\delta_1^n)\xi^{(n-1)} + 4a_nu_x, \\ \phi^{(n)} &\mapsto -4(1-\delta_1^n)\phi^{(n-1)} + 4b_nu_x, \\ \theta^{(n)} &\mapsto -4(1-\delta_1^n)\theta^{(n-1)} + 4c_nu_x, \end{aligned}$$

where $\{a_n, b_n, c_n \in \mathbb{R} : n = 1, 2, \ldots\}$ are arbitrary constants.

PROOF: See Appendix D for details. □

Having identified the nonlocal partial symmetry generators using $\mathcal{R}^{-1}$, the next task is to prolong these vector fields to genuine nonlocal symmetry generators of the prolongation of Section 6.1. If $A\partial_u$ is a nonlocal partial symmetry generator then

$$A\partial_u + \sum_{i=0}^{\infty}(E_i\partial_{p_i} + F_i\partial_{q_i} + G_i\partial_{r_i} + H_i\partial_{s_i})$$

is a true nonlocal symmetry generator if and only if the functions $\{E_i, F_i, G_i, H_i \in \tilde{\mathfrak{A}} : i = 0, 1, \ldots\}$ satisfy the differential equations

$$\begin{aligned}
\tilde{D}_x(E_i) &= F_i, \\
\tilde{D}_x(F_i) &= -2p_iA - 2uE_i - (1-\delta_i^0)E_{i-1}, \\
\tilde{D}_x(G_i) &= H_i, \\
\tilde{D}_x(H_i) &= -2r_iA - 2uG_i - (1-\delta_i^0)G_{i-1},
\end{aligned}$$

together with compatible equations involving total t-derivatives. Further equations,

$$0 = \sum_{i=0}^{n}(s_{n-i}E_i - r_{n-i}F_i - q_{n-i}G_i + p_{n-i}H_i), \quad n = 0, 1, \ldots,$$

arise from the requirement that the algebraic constraints $\Delta^n = 0$ be preserved.

The process begins by prolonging $\xi^{(1)}\partial_u$ to a genuine nonlocal symmetry generator, yielding the vector field which features in Lemma 6.3. All calculations until then are not intended to be rigorous — they motivate the result of that lemma and hopefully suggest ways to repeat this process for other equations of interest. In particular, calculations involving inverting prolonged total derivative operators are performed with gay abandon, despite the reservations discussed in Section 5.3. Nevertheless, the derived vector field is a nonlocal symmetry generator.

For all functions $\{E_i : i = 0, 1, \ldots\}$ the equations

$$(\tilde{D}_x^2 + 2u)(E_i) = -2p_i\xi^{(1)} - (1-\delta_i^0)E_{i-1}, \quad i = 0, 1, \ldots,$$

must be satisfied. The differential operator $\tilde{D}_x^2 + 2u$ admits the factorization

$$\tilde{D}_x^2 + 2u = (\tilde{D}_x - v)(\tilde{D}_x + v) = (p_0^{-1}\tilde{D}_x p_0)(p_0\tilde{D}_x p_0^{-1}) = p_0^{-1}\tilde{D}_x p_0^2\tilde{D}_x p_0^{-1},$$

using, once more, the work of Fordy and Gibbons [30], which allows one to write

$$E_i = -p_0\tilde{D}_x^{-1}p_0^{-2}\tilde{D}_x^{-1}p_0(2p_iA + (1-\delta_i^0)E_{i-1}).$$

Therefore

$$E_0 = -4p_0\tilde{D}_x^{-1}p_0^{-2}\tilde{D}_x^{-1}(p_0^2(p_0s_0 + q_0r_0))$$

and, since

$$\begin{aligned}
\tilde{D}_x^{-1}\left(p_0^2(p_0s_0 + q_0r_0)\right) &= \tilde{D}_x^{-1}(p_0^2(2p_0s_0 - 1)) \\
&= \tilde{D}_x^{-1}(\tfrac{1}{2}p_0^3s_0 + \tfrac{3}{2}p_0^2(1 + q_0r_0) - p_0^2) \\
&= \tilde{D}_x^{-1}(\tfrac{1}{2}p_0^3s_0 + \tfrac{3}{2}p_0^2q_0r_0 + \tfrac{1}{2}p_0^2) \\
&= \tfrac{1}{2}p_0^3r_0 + \tfrac{1}{2}\tilde{D}_x^{-1}(p_0^2),
\end{aligned}$$

it follows that

$$E_0 = -2p_0\tilde{D}_x^{-1}(p_0r_0 + p_0^{-2}\tilde{D}_x^{-1}(p_0^2)).$$

The observation that $\tilde{D}_x(p_0^{-1}r_0) = p_0^{-2}$ leads to

$$\begin{aligned} E_0 &= -2p_0\tilde{D}_x^{-1}((p_0^{-1}r_0)p_0^2 + p_0^{-2}\tilde{D}_x^{-1}(p_0^2)) \\ &= -2r_0\tilde{D}_x^{-1}(p_0^2) \end{aligned}$$

and the nonlocal partial symmetry generator $\xi^{(1)}\partial_u$ has been prolonged as far as the ∂_{p_0} term. From equation (6.12) it is clear that no new nonlocal variables need to be introduced to calculate E_0, which now takes the form

$$E_0 = -2r_0(p_1q_0 - p_0q_1).$$

This process can be continued, and yields

$$\begin{aligned} E_1 &= -p_0\tilde{D}_x^{-1}p_0^{-2}\tilde{D}_x^{-1}p_0(4p_1(p_0s_0 + q_0r_0) - 2r_0(p_1q_0 - p_0q_1)) \\ &= -p_0\tilde{D}_x^{-1}p_0^{-2}\tilde{D}_x^{-1}(4p_0q_0p_1r_0 + 2p_0^2q_1r_0 + 2p_0^2p_1s_0 + 2p_0p_1(p_0s_0 - q_0r_0)) \\ &= -p_0\tilde{D}_x^{-1}p_0^{-2}(2p_0^2p_1r_0 + 2\tilde{D}_x^{-1}(p_0p_1)) \\ &= -p_0\tilde{D}_x^{-1}(2(p_0^{-1}r_0)(p_0p_1) + 2p_0^{-2}\tilde{D}_x^{-1}(p_0p_1)) \\ E_1 &= -2r_0\tilde{D}_x^{-1}(p_0p_1). \end{aligned}$$

This suggests that

$$E_i = -2r_0\tilde{D}_x^{-1}(p_0p_i), \quad i = 0, 1, \ldots$$

and, using a symmetry argument, that

$$G_i = -2p_0\tilde{D}_x^{-1}(r_0r_i), \quad i = 0, 1, \ldots .$$

Abbreviations are now introduced for some polynomials involving the pseudopotentials which occur frequently. For each $m, n = 0, 1, \ldots$ let

$$\begin{aligned} \mathsf{D}^{-1}(p_mp_n) &= \sum_{i=0}^{\min\{m,n\}} (p_{m+n+1-i}q_i - p_iq_{m+n+1-i}), \\ \mathsf{D}^{-1}(r_mr_n) &= \sum_{i=0}^{\min\{m,n\}} (r_{m+n+1-i}s_i - r_is_{m+n+1-i}), \\ \mathsf{D}^{-1}(p_mr_n + p_nr_m) &= \sum_{i=0}^{\min\{m,n\}} (p_{m+n+1-i}s_i - p_is_{m+n+1-i} \\ &\qquad + q_ir_{m+n+1-i} - q_{m+n+1-i}r_i). \end{aligned}$$

The following lemma explains why this particular notation has been used.

Lemma 6.2 *For all* $m, n = 0, 1, \ldots$

$$\begin{aligned} \tilde{D}_x\left(\mathsf{D}^{-1}(p_m p_n)\right) &= p_m p_n, \\ \tilde{D}_x\left(\mathsf{D}^{-1}(r_m r_n)\right) &= r_m r_n, \\ \tilde{D}_x\left(\mathsf{D}^{-1}(p_m r_n + p_n r_m)\right) &= p_m r_n + p_n r_m. \end{aligned}$$

PROOF: Straightforward. □

The prolongation of $\xi^{(1)}\partial_u$ to a nonlocal symmetry generator now takes a particularly simple form.

Lemma 6.3 *The vector field*

$$\mathbf{u}_{-1} = \xi^{(1)}\partial_u + \sum_{i=0}^{\infty}\left(A_i^{(1)}\partial_{p_i} + \tilde{D}_x(A_i^{(1)})\partial_{q_i} + B_i^{(1)}\partial_{r_i} + \tilde{D}_x(B_i^{(1)})\partial_{s_i}\right)$$

with

$$A_i^{(1)} = -2r_0\mathsf{D}^{-1}(p_0 p_i), \quad B_i^{(1)} = -2p_0\mathsf{D}^{-1}(r_0 r_i), \quad i = 0, 1, \ldots\ ,$$

is a nonlocal symmetry generator of the prolongation of the KdV equation described by equations (6.4) *and the algebraic constraints* $\{\Delta^n = 0 : n = 0, 1, \ldots\}$.
PROOF: See Appendix D for details. □

The computation of the commutators of $\mathbf{u}_{-1}$ with suitable internal symmetry generators proves to be a reasonably efficient way of prolonging the other nonlocal partial symmetry generators of Theorem 6.1 to genuine nonlocal symmetry generators.

Since the commutator of two symmetry generators is itself a symmetry generator, the vector field $\mathbf{w}_{-1} = \frac{1}{2}[\mathbf{u}_{-1}, \mathbf{w}_0]$ must be a symmetry generator of the prolongation of the KdV equation being considered here. The coefficient of ∂_u of this vector field is

$$du(\mathbf{w}_{-1}) = -\tfrac{1}{2}\mathbf{w}_0(\xi^{(1)}) = -2p_0 q_0 = \theta^{(1)},$$

so that $\mathbf{w}_{-1}$ is equivalent to the nonlocal partial symmetry generator $\theta^{(1)}\partial_u$. Straightforward calculations confirm that

$$dp_i(\mathbf{w}_{-1}) = -\tfrac{1}{2}\mathbf{w}_0(A_i^{(1)}) = p_0(p_{i+1}q_0 - p_0 q_{i+1}) = p_0\mathsf{D}^{-1}(p_0 p_i)$$

and

$$\begin{aligned} dr_i(\mathbf{w}_{-1}) &= \tfrac{1}{2}\mathbf{u}_{-1}(p_i) - \tfrac{1}{2}\mathbf{w}_0(B_i^{(1)}) \\ &= \tfrac{1}{2}A_i^{(1)} + p_0(p_{i+1}s_0 - p_0 s_{i+1} + q_0 r_{i+1} - q_{i+1}r_0) \\ dr_i(\mathbf{w}_{-1}) &= p_0\mathsf{D}^{-1}(p_0 r_i + p_i r_0) - r_0\mathsf{D}^{-1}(p_0 p_i). \end{aligned}$$

This procedure can be repeated by introducing the vector field $\mathbf{v}_{-1} = \frac{1}{2}[\mathbf{v}_0, \mathbf{u}_{-1}]$ which is calculated with similar ease. The two new nonlocal symmetry generators are now described formally.

Lemma 6.4 *The vector fields*

$$\mathbf{v}_{-1} = \phi^{(1)}\partial_u + \sum_{i=0}^{\infty}\left(E_i^{(1)}\partial_{p_i} + \tilde{D}_x(E_i^{(1)})\partial_{q_i} + F_i^{(1)}\partial_{r_i} + \tilde{D}_x(F_i^{(1)})\partial_{s_i}\right),$$

$$\mathbf{w}_{-1} = \theta^{(1)}\partial_u + \sum_{i=0}^{\infty}\left(G_i^{(1)}\partial_{p_i} + \tilde{D}_x(G_i^{(1)})\partial_{q_i} + H_i^{(1)}\partial_{r_i} + \tilde{D}_x(H_i^{(1)})\partial_{s_i}\right),$$

with

$$\begin{aligned} E_i^{(1)} &= r_0\mathsf{D}^{-1}(p_0 r_i + p_i r_0) - p_0\mathsf{D}^{-1}(r_0 r_i), \\ F_i^{(1)} &= r_0\mathsf{D}^{-1}(r_0 r_i), \\ G_i^{(1)} &= p_0\mathsf{D}^{-1}(p_0 p_i), \\ H_i^{(1)} &= p_0\mathsf{D}^{-1}(p_0 r_i + p_i r_0) - r_0\mathsf{D}^{-1}(p_0 p_i), \end{aligned}$$

are nonlocal symmetry generators of the prolongation of the KdV equation described by equations (6.4) *and the algebraic constraints* $\{\Delta^n = 0 : n = 0, 1, \ldots\}$.

PROOF: See Appendix D for details. □

Thus, three of the nonlocal partial symmetry generators of Theorem 6.1 have been prolonged to true nonlocal symmetry generators. The Lie algebra generated by $\{\mathbf{u}_{-1}, \mathbf{v}_{-1}, \mathbf{w}_{-1}\}$ yields genuine symmetry generators equivalent to the nonlocal partial symmetry generators of that theorem. Evaluating the generator $\mathbf{u}_{-2} = [\mathbf{v}_{-1}, \mathbf{w}_{-1}]$ uncovers the form of these nonlocal symmetry generators. The coefficient of ∂_u in this vector field is $\xi^{(2)}$ and, after much tedious calculation, that of ∂_{p_i} is found to be

$$\begin{aligned} dp_i(\mathbf{u}_{-2}) &= \mathbf{v}_{-1}(G_i^{(1)}) - \mathbf{w}_{-1}(E_i^{(1)}) \\ &= -2r_0\mathsf{D}^{-1}(p_i p_1) - 2r_1\mathsf{D}^{-1}(p_i p_0) \\ dp_i(\mathbf{u}_{-2}) &= -2\sum_{j=0}^{1} r_j\mathsf{D}^{-1}(p_i p_{1-j}). \end{aligned}$$

Comparing this expression with the corresponding coefficient in $\mathbf{u}_{-1}$, and calculating other coefficients if necessary, one is led to postulate the prolongations of $\xi^{(n)}\partial_u$, $\phi^{(n)}\partial_u$ and $\theta^{(n)}\partial_u$ which feature in the following theorem.

Theorem 6.5 *The vector fields*

$$\mathbf{u}_{-n} = \xi^{(n)}\partial_u + \sum_{i=0}^{\infty}\left(A_i^{(n)}\partial_{p_i} + \tilde{D}_x(A_i^{(n)})\partial_{q_i} + B_i^{(n)}\partial_{r_i} + \tilde{D}_x(B_i^{(n)})\partial_{s_i}\right),$$
$$\mathbf{v}_{-n} = \phi^{(n)}\partial_u + \sum_{i=0}^{\infty}\left(E_i^{(n)}\partial_{p_i} + \tilde{D}_x(E_i^{(n)})\partial_{q_i} + F_i^{(n)}\partial_{r_i} + \tilde{D}_x(F_i^{(n)})\partial_{s_i}\right),$$
$$\mathbf{w}_{-n} = \theta^{(n)}\partial_u + \sum_{i=0}^{\infty}\left(G_i^{(n)}\partial_{p_i} + \tilde{D}_x(G_i^{(n)})\partial_{q_i} + H_i^{(n)}\partial_{r_i} + \tilde{D}_x(H_i^{(n)})\partial_{s_i}\right),$$

$n = 1, 2, \ldots$, *with*

$$\begin{aligned}
A_i^{(n)} &= -2\sum_{j=0}^{n-1} r_j \mathsf{D}^{-1}(p_i p_{n-1-j}),\\
B_i^{(n)} &= -2\sum_{j=0}^{n-1} p_j \mathsf{D}^{-1}(r_i r_{n-1-j}),\\
E_i^{(n)} &= \sum_{j=0}^{n-1}\left(r_j \mathsf{D}^{-1}(p_i r_{n-1-j} + p_{n-1-j} r_i) - p_j \mathsf{D}^{-1}(r_i r_{n-1-j})\right),\\
F_i^{(n)} &= \sum_{j=0}^{n-1} r_j \mathsf{D}^{-1}(r_i r_{n-1-j}),\\
G_i^{(n)} &= \sum_{j=0}^{n-1} p_j \mathsf{D}^{-1}(p_i p_{n-1-j}),\\
H_i^{(n)} &= \sum_{j=0}^{n-1}\left(p_j \mathsf{D}^{-1}(p_i r_{n-1-j} + p_{n-1-j} r_i) - r_j \mathsf{D}^{-1}(p_i p_{n-1-j})\right),
\end{aligned}$$

are nonlocal symmetry generators of the prolongation of the KdV equation described by equations (6.4) and the algebraic constraints $\{\Delta^n = 0 : n = 0, 1, \ldots\}$. The Lie algebra with basis $\{\mathbf{u}_n, \mathbf{v}_n, \mathbf{w}_n : n \in \mathbb{Z}\}$ is isomorphic to $\mathfrak{sl}(2,\mathbb{R}) \otimes \mathbb{R}[\lambda, \lambda^{-1}]$ and has commutator table

	$\mathbf{u}_n$	$\mathbf{v}_n$	$\mathbf{w}_n$
$\mathbf{u}_m$	0	$-2\mathbf{v}_{m+n}$	$2\mathbf{w}_{m+n}$
$\mathbf{v}_m$		0	$\mathbf{u}_{m+n}$
$\mathbf{w}_m$			0

PROOF: See Appendix D for details. □

This theorem has uncovered the remarkable fact that whereas $\mathcal{R}$ acts on zero to generate an infinite-dimensional, Abelian Lie algebra of generalized symmetries for the KdV equation, $\mathcal{R}^{-1}$ acts on zero to generate an infinite-dimensional, non-Abelian Lie algebra of nonlocal symmetries. Furthermore, this is a subalgebra of the familiar loop algebra over $\mathfrak{sl}(2,\mathbb{R})$, a structure which appears frequently in soliton theory.

Let $\mathcal{L}^{\text{KdV}}_{\text{Nonlocal}}$ denote the vector space of nonlocal symmetry generators with basis $\{\mathbf{u}_{-n}, \mathbf{v}_{-n}, \mathbf{w}_{-n} : n = 1, 2, \ldots\}$. It follows from Theorem 6.5 that $\mathcal{L}^{\text{KdV}}_{\text{Nonlocal}}$ is closed under the Lie bracket and

$$\mathfrak{sl}(2,\mathbb{R}) \otimes \mathbb{R}[\lambda, \lambda^{-1}] = \mathcal{L}^{\text{KdV}}_{\text{Internal}} + \mathcal{L}^{\text{KdV}}_{\text{Nonlocal}}$$

splits the loop algebra over $\mathfrak{sl}(2,\mathbb{R})$ into the *vector space* direct sum of two *subalgebras*. This is not an algebraic direct sum since the spaces of internal and nonlocal symmetry generators do not commute in this example.

6.3 Symmetries of related equations

Differential equations related to the KdV equation, such as the mKdV and PmKdV equations, admit a similarly rich set of nonlocal symmetry generators. As is the case for the KdV equation, these nonlocal symmetries can be constructed by inverting recursion operators for each differential equation. The results are more easily obtained by using Section 6.1 to construct an appropriate infinite-dimensional prolongation for each equation in turn, and then interpreting Theorem 6.5 in light of these new prolongations.

When factorizing the recursion operator for the KdV equation in Section 6.2 the variable $v = -p_0^{-1} q_0$ was found to satisfy the system of Riccati equations (6.8). By eliminating u between these equations one finds that v is a solution to the mKdV equation

$$0 = v_t + v_{xxx} - 6v^2 v_x.$$

This observation can be used to construct a prolongation of the mKdV equation analogous to the one derived in Section 6.1 for the KdV equation.

Begin by eliminating all appearances of q_0 from equations (6.4) using $q_0 = -vp_0$, to obtain equations for v,

$$v_x = v^2 + 2u, \tag{6.13}$$

$$v_t = -4uv^2 - 4u_x v - 2(u_{xx} + 4u^2), \tag{6.14}$$

together with prolongation equations for the remaining pseudopotentials. The variable u is eliminated next, using equation (6.13), leaving a system of equations which define the first order derivatives of the remaining pseudopotentials as appropriate functions of v and its derivatives together with the pseudopotentials themselves. This does not, strictly, describe a Wahlquist-Estabrook prolongation of the mKdV equation because of prolongation equations such as

$$\begin{aligned} s_{0,x} &= (v^2 - v_x)r_0, \\ s_{0,t} &= (v_{xxx} - 2vv_{xx} - 4v^2 v_x + 2v^4)r_0 + (2vv_x - v_{xx})s_0. \end{aligned}$$

Since the mKdV equation is of third order, the definition of Wahlquist-Estabrook prolongations requires that the expressions on the right hand side of the two equations above can only depend on derivatives of v up to second order. There are several ways to avoid this problem. Because the only forbidden terms involve v_{xxx}, this variable could be eliminated using $v_{xxx} = 6v^2 v_x - v_t$, yielding equations of the appropriate form. Alternatively, one could leave the prolongation equations as they are and treat them as determining a valid Wahlquist-Estabrook prolongation of the fourth order system comprising the mKdV equation and its first order differential consequences.

The preferred alternative involves making an invertible change of coordinates. Because the appearance of v_{xxx} in the temporal prolongation equations is closely related to the appearance of v_x in the spatial equations, one attempts to eliminate v_x from the spatial equations. For instance,

$$(vr_0 + s_0)_x = v(vr_0 + s_0),$$

with the right hand side not involving v_x explicitly. Similarly, the equation for $(vr_0 + s_0)_t$ does not involve v_{xxx} explicitly. Analysis of the complete set of prolongation equations motivates the invertible change of coordinates from $\{p_m, q_m, r_m, s_m : m = 0, 1, \ldots\}$ to $\{v, \tilde{p}_0, \tilde{r}_0, \tilde{s}_0, \tilde{p}_n, \tilde{q}_n, \tilde{r}_n, \tilde{s}_n : n = 1, 2, \ldots\}$ which is described by

$$p_m = \tilde{p}_m, \quad q_m = -v\tilde{p}_m + (1 - \delta_m^0)\tilde{q}_m, \quad r_m = \tilde{r}_m, \quad s_m = -v\tilde{r}_m + \tilde{s}_m, \tag{6.15}$$

for $m = 0, 1, \ldots$. In terms of these coordinates, equations (6.4) give the equations determining v, equations (6.13) and (6.14), together with the system of equations

$$
\begin{aligned}
\tilde{p}_{m,x} &= -v\tilde{p}_m + (1-\delta^0_m)\tilde{q}_m, \\
\tilde{q}_{n,x} &= -\tilde{p}_{n-1} + v\tilde{q}_n, \\
\tilde{r}_{m,x} &= -v\tilde{r}_m + \tilde{s}_m, \\
\tilde{s}_{m,x} &= v\tilde{s}_m - (1-\delta^0_m)\tilde{r}_{m-1}, \\
\tilde{p}_{m,t} &= (v_{xx} - 2v^3)\tilde{p}_m - 2(1-\delta^0_m)(2v\tilde{p}_{m-1} + (v_x - v^2)\tilde{q}_m) \\
&\quad + 4(1-\delta^0_m)(1-\delta^1_m)\tilde{q}_{m-1}, \\
\tilde{q}_{n,t} &= -2(v_x + v^2)\tilde{p}_{n-1} - (v_{xx} - 2v^3)\tilde{q}_n - 4(1-\delta^1_n)(\tilde{p}_{n-2} + v\tilde{q}_{n-1}), \\
\tilde{r}_{m,t} &= (v_{xx} - 2v^3)\tilde{r}_m - 2(v_x - v^2)\tilde{s}_m + 4(1-\delta^0_m)(-v\tilde{r}_{m-1} + \tilde{s}_{m-1}), \\
\tilde{s}_{m,t} &= -(v_{xx} - 2v^3)\tilde{s}_m + 2(1-\delta^0_m)(-(v_x + v^2)\tilde{r}_{m-1} + 2v\tilde{s}_{m-1}) \\
&\quad - 4(1-\delta^0_m)(1-\delta^1_m)\tilde{r}_{m-2},
\end{aligned}
\tag{6.16}
$$

where $m = 0, 1, \ldots$ and $n = 1, 2, \ldots$. The algebraic constraints become

$$
\begin{aligned}
1 &= \tilde{p}_0\tilde{s}_0, \\
0 &= \tilde{p}_n\tilde{s}_0 + \sum_{i=0}^{n-1}(\tilde{p}_i\tilde{s}_{n-i} - \tilde{q}_{n-i}\tilde{r}_i), \quad n = 1, 2, \ldots ,
\end{aligned}
$$

and combine with equations (6.16) to describe a generalized Wahlquist-Estabrook prolongation of the mKdV equation.

The loop algebra of symmetries of the KdV equation which was presented in Theorem 6.5 immediately yields a loop algebra of symmetries of this prolongation of the mKdV equation. The generators of this algebra can be found by writing the vector fields of Theorem 6.5 in terms of the new coordinates introduced above. The results of this calculation will not be given here, because the main point of interest is the manner in which the resulting loop algebra splits into internal and nonlocal symmetries. Therefore, all that will be calculated are the coefficients of ∂_v in the expansions of the vector fields in the new coordinates. This is most easily done by evaluating $\mathbf{u}(-p_0^{-1}q_0)$ for each vector field $\mathbf{u}$ of Theorem 6.5. All internal symmetry generators $\{\mathbf{u}_m, \mathbf{v}_m, \mathbf{w}_m : m = 0, 1, \ldots\}$ presented in Section 6.1 leave $v = -p_0^{-1}q_0$ invariant, with the exception of $\mathbf{v}_0$, which yields $\mathbf{v}_0(-p_0^{-1}q_0) = \tilde{p}_0^{-2}$ when the algebraic constraints $\{\Delta^n = 0 : n = 0, 1, \ldots\}$ are satisfied. Furthermore, for each $n = 1, 2, \ldots$

$$
\begin{aligned}
\tilde{\xi}^{(n)} &= \mathbf{u}_{-n}(-p_0^{-1}q_0) \\
&= p_0^{-2}q_0A_0^{(n)} - p_0^{-1}\tilde{D}_x(A_0^{(n)}) \\
&= 2\sum_{j=0}^{n-1}(\tilde{p}_{n-1-j}\tilde{r}_j - \tilde{q}_{n-j}\tilde{s}_j)
\end{aligned}
$$

in terms of the new coordinates, and similarly,

$$
\begin{aligned}
\tilde{\phi}^{(n)} &= \mathbf{v}_{-n}(-p_0^{-1}q_0) &&= -\sum_{j=0}^{n-1}\tilde{r}_j\tilde{r}_{n-1-j} + \sum_{j=0}^{n}\tilde{s}_j\tilde{s}_{n-j}, \\
\tilde{\theta}^{(n)} &= \mathbf{w}_{-n}(-p_0^{-1}q_0) &&= -\sum_{j=0}^{n-1}\tilde{p}_j\tilde{p}_{n-1-j} + \sum_{j=1}^{n-1}\tilde{q}_j\tilde{q}_{n-j}.
\end{aligned}
$$

Thus, all of the nonlocal symmetry generators of Section 6.2 are also nonlocal symmetry generators of the mKdV equation.

Let $\mathcal{L}^{\mathrm{mKdV}}_{\mathrm{Internal}}$, with basis $\{\mathbf{u}_0, \mathbf{w}_0, \mathbf{u}_n, \mathbf{v}_n, \mathbf{w}_n : n = 1, 2, \ldots\}$, and $\mathcal{L}^{\mathrm{mKdV}}_{\mathrm{Nonlocal}}$, with basis $\{\mathbf{v}_0, \mathbf{u}_{-n}, \mathbf{v}_{-n}, \mathbf{w}_{-n} : n = 1, 2, \ldots\}$, denote subspaces of internal and nonlocal symmetry generators, respectively, of this prolongation. Then

$$
\mathfrak{sl}(2, \mathbb{R}) \otimes \mathbb{R}[\lambda, \lambda^{-1}] = \mathcal{L}^{\mathrm{mKdV}}_{\mathrm{Internal}} + \mathcal{L}^{\mathrm{mKdV}}_{\mathrm{Nonlocal}}
$$

splits the loop algebra over $\mathfrak{sl}(2, \mathbb{R})$ into the vector space direct sum of two subalgebras. All that has changed in going from the KdV to mKdV equations is the manner in which the loop algebra splits. Essentially, $\mathbf{v}_0$ has moved from the subalgebra of internal symmetry generators to that of nonlocal ones. Notice that $\mathbf{v}_0$ and $\mathbf{w}_{-1}$ are equivalent to the two nonlocal partial symmetry generators of the mKdV equation found in Example 5.9. The observation that these two vector fields generate the infinite-dimensional algebra $\mathcal{L}^{\mathrm{mKdV}}_{\mathrm{Nonlocal}}$ confirms the claim made after that example that any prolongation admitting true nonlocal symmetry generators equivalent to these two must admit infinitely many such generators. Furthermore, note that $\mathbf{v}_0$, $\mathbf{u}_{-1}$ and $\mathbf{w}_{-1}$ correspond to the three nonlocal partial symmetry generators of the mKdV equation found in Example 5.18 by evaluating $\mathcal{R}^{-1}(0)$. In fact, applying the recursion operator of the mKdV equation to the symmetry characteristics $\tilde{\xi}^{(n)}$, $\tilde{\theta}^{(n)}$ and $\tilde{\phi}^{(n)}$ confirms that the inverse of that recursion operator acts on zero to generate $\mathcal{L}^{\mathrm{mKdV}}_{\mathrm{Nonlocal}}$, as claimed at the beginning of this section.

This phenomenon continues as one considers other equations related to the KdV and mKdV equations. For example, the variable $w = -\log \tilde{p}_0$ satisfies the PmKdV equation

$$0 = w_t + w_{xxx} - 2w_x^3.$$

As before, one can introduce an invertible change of coordinates transforming the prolongation of the mKdV equation into an infinite-dimensional prolongation of the PmKdV equation. The pseudopotentials $\{\tilde{p}_m, \tilde{q}_n, \tilde{r}_m, \tilde{s}_m : m = 0, 1, \ldots, n = 1, 2, \ldots\}$ of the mKdV equation are replaced by $\{w, p'_n, q'_n, r'_m, s'_n : m = 0, 1, \ldots, n = 1, 2, \ldots\}$ via the transformation

$$\begin{aligned} \tilde{p}_m &= e^{-w}(\delta_m^0 + (1 - \delta_m^0)p'_m), \\ \tilde{q}_n &= e^{w}q'_n, \\ \tilde{r}_m &= e^{-w}r'_m, \\ \tilde{s}_m &= e^{w}(\delta_m^0 + (1 - \delta_m^0)s'_m), \end{aligned} \tag{6.17}$$

where $m = 0, 1, \ldots$ and $n = 1, 2, \ldots$. Equations (6.16) yield a system of equations defining w,

$$w_x = v, \quad w_t = -v_{xx} + 2v^3, \tag{6.18}$$

and relating the mKdV and PmKdV equations, as well as an infinite-dimensional system describing a prolongation of the PmKdV equation. The prolongation equations are

$$\begin{aligned} p'_{n,x} &= e^{2w}q'_n, \\ q'_{n,x} &= -e^{-2w}(\delta_n^1 + (1 - \delta_n^1)p'_{n-1}), \\ r'_{m,x} &= e^{2w}(\delta_m^0 + (1 - \delta_m^0)s'_m), \\ s'_{n,x} &= -e^{-2w}r'_{n-1}, \\ p'_{n,t} &= -2(w_{xx} - w_x^2)e^{2w}q'_n - 4w_x(\delta_n^1 + (1 - \delta_n^1)p'_{n-1}) \\ &\quad + 4(1 - \delta_n^1)e^{2w}q'_{n-1}, \\ q'_{n,t} &= -2(w_{xx} + w_x^2)e^{-2w}(\delta_n^1 + (1 - \delta_n^1)p'_{n-1}) + 4(1 - \delta_n^1)w_x q'_{n-1} \\ &\quad - 4(1 - \delta_n^1)e^{-2w}(\delta_n^2 + (1 - \delta_n^2)p'_{n-2}), \\ r'_{m,t} &= -2(w_{xx} - w_x^2)e^{2w}(\delta_m^0 + (1 - \delta_m^0)s'_m) - 4(1 - \delta_m^0)w_x r'_{m-1} \\ &\quad + 4(1 - \delta_m^0)e^{2w}(\delta_m^1 + (1 - \delta_m^1)s'_{m-1}), \end{aligned} \tag{6.19}$$

$$\begin{aligned} s'_{n,t} &= -2(w_{xx}+w_x^2)e^{-2w}r'_{n-1}+4w_x(\delta_n^1+(1-\delta_n^1)s'_{n-1}) \\ &\quad -4(1-\delta_n^1)e^{-2w}r'_{n-2}, \end{aligned}$$

where $m=0,1,\dots$ and $n=1,2,\dots$. Algebraic constraints take the form

$$0 = s'_n + \sum_{i=1}^{n-1} p'_i s'_{n-i} + p'_n - \sum_{i=0}^{n-1} q'_{n-i} r'_i, \quad n=1,2,\dots .$$

As was the case for the mKdV equation, Theorem 6.5 yields a loop algebra of symmetries of this prolongation of the PmKdV equation. For each vector field $\mathbf{u}$ in that theorem, its nature with respect to this prolongation is determined by $\mathbf{u}(w) = \mathbf{u}(-\log p_0)$. The vectors $\{\mathbf{w}_0, \mathbf{u}_n, \mathbf{v}_n, \mathbf{w}_n : n=1,2,\dots\}$ span the subalgebra $\mathcal{L}^{\text{PmKdV}}_{\text{Internal}}$ of symmetry generators leaving w invariant and, since $\mathbf{u}_0(w)=1$, $\mathbf{u}_0$ is simply an inherited symmetry generator from the PmKdV equation. As such, $\mathcal{L}^{\text{PmKdV}}_{\text{Inherited}} = \text{sp}\{\mathbf{u}_0\}$ is introduced. It denotes the elements of the loop algebra which are inherited, but not internal, symmetries of the prolonged system. Finally, $\{\mathbf{v}_0, \mathbf{u}_{-n}, \mathbf{v}_{-n}, \mathbf{w}_{-n} : n=1,2,\dots\}$ are all nonlocal symmetry generators, spanning the subalgebra $\mathcal{L}^{\text{PmKdV}}_{\text{Nonlocal}}$. Another splitting of the loop algebra results, this time into

$$\mathfrak{sl}(2,\mathbb{R}) \otimes \mathbb{R}[\lambda,\lambda^{-1}] = \mathcal{L}^{\text{PmKdV}}_{\text{Internal}} + \mathcal{L}^{\text{PmKdV}}_{\text{Inherited}} + \mathcal{L}^{\text{PmKdV}}_{\text{Nonlocal}},$$

which is a vector space direct sum with three component subalgebras.

The final equation considered is the PPmKdV equation

$$0 = y_t + y_{xxx} - \tfrac{3}{2} y_x^{-1} y_{xx}^2$$

satisfied by the variable $y = r'_0$. It does not seem to be possible to change coordinates so as to eliminate y_x from the spatial equations for the resulting prolongation of the PPmKdV equation, so instead $\{p'_n, q'_n, r'_n, s'_n : n=1,2,\dots\}$ are left, unchanged, as pseudopotentials for this equation. The natures of the vector fields $\mathbf{u}$ of Theorem 6.5 are determined by $\mathbf{u}(y) = \mathbf{u}(-p_0^{-1}r_0)$. Thus,

$$\mathbf{u}_0 = -2y\partial_y + \dots, \quad \mathbf{v}_0 = y^2\partial_y + \dots, \quad \mathbf{w}_0 = \partial_y + \dots,$$

so let $\mathcal{L}^{\text{PPmKdV}}_{\text{Inherited}}$ denote the symmetry algebra with basis $\{\mathbf{u}_0, \mathbf{v}_0, \mathbf{w}_0\}$. Furthermore, internal symmetry generators are $\{\mathbf{u}_n, \mathbf{v}_n, \mathbf{w}_n : n=1,2,\dots\}$ and span $\mathcal{L}^{\text{PPmKdV}}_{\text{Internal}}$, while nonlocal symmetry generators are $\{\mathbf{u}_{-n}, \mathbf{v}_{-n}, \mathbf{w}_{-n} : n=1,2,\dots\}$, spanning

$\mathcal{L}^{\text{PPmKdV}}_{\text{Nonlocal}}$. Once more the loop algebra splits into the vector space direct sum of three subalgebras:

$$\mathfrak{sl}(2,\mathbb{R}) \otimes \mathbb{R}[\lambda, \lambda^{-1}] = \mathcal{L}^{\text{PPmKdV}}_{\text{Internal}} + \mathcal{L}^{\text{PPmKdV}}_{\text{Inherited}} + \mathcal{L}^{\text{PPmKdV}}_{\text{Nonlocal}}.$$

Thus, for each of the KdV, mKdV, PmKdV and PPmKdV equations, there exists an infinite-dimensional Wahlquist-Estabrook prolongation admitting a loop algebra of inherited, internal and nonlocal symmetry generators. Loop algebras have usually appeared in the study of differential equations in an abstract setting, such as the prolongation algebras of Wahlquist and Estabrook [21], [42], [43]. Here, however, these algebras are realized in terms of symmetry generators. Shown schematically in Figure 6.1 are the loop algebras of symmetries found for each of the four equations studied in this chapter. The figure demonstrates the various ways in which these al-

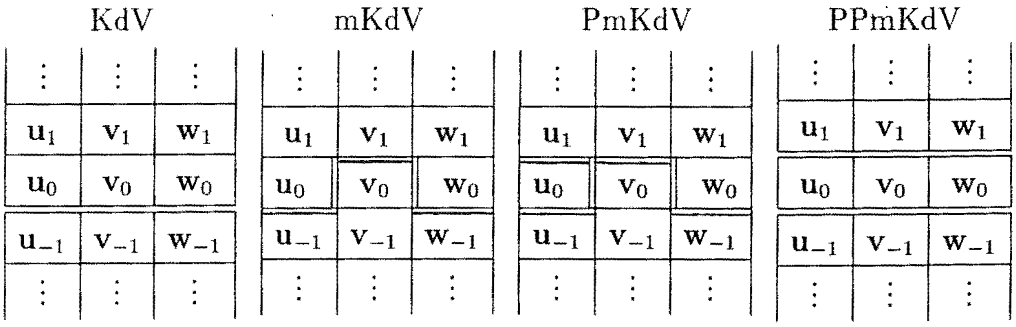

Figure 6.1: Splittings of loop algebras of symmetries for various integrable equations

gebras split into vector space direct sums of inherited, internal and nonlocal symmetry subalgebras. For each equation the subalgebra of internal symmetry generators is shown in the top part of the diagram and the subalgebra of nonlocal symmetry generators appears in the bottom part. In the case of the PmKdV and PPmKdV equations the isolated finite-dimensional subalgebras represent the inherited symmetry generators in the loop algebra, which are not internal to the prolongation.

This chapter concludes with a brief discussion on the origins of the prolongations of the mKdV, PmKdV and PPmKdV equations used here and on their relationship to the one presented in Section 6.1. Recall that one derivation of the system of equations (6.4) involved the linear system

$$\mathbf{\Psi}_x = \mathbf{\Psi}\mathbf{A}, \quad \mathbf{\Psi}_t = \mathbf{\Psi}\mathbf{B},$$

where $\mathbf{A}$ and $\mathbf{B}$ are the $\mathfrak{sl}(2,\mathbb{R})\otimes\mathbb{R}[\lambda]$-valued functions given by equations (6.6) and

$$\mathbf{\Psi} = \sum_{n=0}^{\infty} \lambda^n \mathbf{\Psi_n}$$

is described by

$$\mathbf{\Psi}_n = \begin{pmatrix} p_n & q_n \\ r_n & s_n \end{pmatrix},$$

with $\det \mathbf{\Psi}$ identically equal to one. Let $\tilde{\mathbf{\Psi}}$ be the $SL(2,\mathbb{R})$-valued function

$$\tilde{\mathbf{\Psi}} = \sum_{n=0}^{\infty} \lambda^n \tilde{\mathbf{\Psi}}_n = \begin{pmatrix} \tilde{p}_0 & 0 \\ \tilde{r}_0 & \tilde{s}_0 \end{pmatrix} + \sum_{n=1}^{\infty} \lambda^n \begin{pmatrix} \tilde{p}_n & \tilde{q}_n \\ \tilde{r}_n & \tilde{s}_n \end{pmatrix}.$$

The change of coordinates given by equations (6.15) is then just $\tilde{\mathbf{\Psi}} = \mathbf{\Psi}\mathbf{V}$, where

$$\mathbf{V} = \begin{pmatrix} 1 & v \\ 0 & 1 \end{pmatrix},$$

so that the coordinate change is a gauge transformation. Furthermore, the transformed linear system is

$$\tilde{\mathbf{\Psi}}_x = \tilde{\mathbf{\Psi}}\tilde{\mathbf{A}}, \quad \tilde{\mathbf{\Psi}}_t = \tilde{\mathbf{\Psi}}\tilde{\mathbf{B}}, \tag{6.20}$$

where

$$\tilde{\mathbf{A}} = \mathbf{V}^{-1}\mathbf{A}\mathbf{V} + \mathbf{V}^{-1}\mathbf{V}_x = \begin{pmatrix} -v & v_x - v^2 - 2u \\ 1 & v \end{pmatrix} + \lambda \begin{pmatrix} 0 & -4 \\ 0 & 0 \end{pmatrix}$$

and

$$\begin{aligned} \tilde{\mathbf{B}} &= \mathbf{V}^{-1}\mathbf{B}\mathbf{V} + \mathbf{V}^{-1}\mathbf{V}_t \\ &= \begin{pmatrix} 4uv + 2u_x & v_t + 4uv^2 + 4u_x v + 2u_{xx} + 8u^2 \\ -4u & -4uv - 2u_x \end{pmatrix} \\ &\quad + \lambda \begin{pmatrix} -4v & -4v^2 - 4u \\ 4 & 4v \end{pmatrix} + \lambda^2 \begin{pmatrix} 0 & -4 \\ 0 & 0 \end{pmatrix}. \end{aligned}$$

The coefficients of powers of λ in equations (6.20) yield equations (6.13) and (6.14), which determine v, together with the prolongation equations (6.16). The algebraic constraints arise from forcing $\det \tilde{\mathbf{\Psi}} = 1$ for all λ.

A similar interpretation applies for the coordinate change leading to the PmKdV equation. This time the $SL(2,\mathbb{R})$-valued function is

$$\mathbf{\Psi}' = \sum_{n=0}^{\infty} \lambda^n \mathbf{\Psi}'_n = \begin{pmatrix} 1 & 0 \\ r'_0 & 1 \end{pmatrix} + \sum_{n=1}^{\infty} \lambda^n \begin{pmatrix} p'_n & q'_n \\ r'_n & s'_n \end{pmatrix}$$

and

$$\mathbf{W} = \begin{pmatrix} e^w & 0 \\ 0 & e^{-w} \end{pmatrix}.$$

The coordinate change, equations (6.17), corresponds to the gauge transformation $\mathbf{\Psi}' = \tilde{\mathbf{\Psi}}\mathbf{W}$ and the transformed linear system is

$$\mathbf{\Psi}'_x = \mathbf{\Psi}'\mathbf{A}', \quad \mathbf{\Psi}'_t = \mathbf{\Psi}'\mathbf{B}', \tag{6.21}$$

where

$$\mathbf{A}' = \mathbf{W}^{-1}\tilde{\mathbf{A}}\mathbf{W} + \mathbf{W}^{-1}\mathbf{W}_x = \begin{pmatrix} w_x - v & 0 \\ e^{2w} & -w_x + v \end{pmatrix} + \lambda \begin{pmatrix} 0 & -e^{-2w} \\ 0 & 0 \end{pmatrix}$$

and

$$\begin{aligned} \mathbf{B}' &= \mathbf{W}^{-1}\tilde{\mathbf{B}}\mathbf{W} + \mathbf{W}^{-1}\mathbf{W}_t \\ &= \begin{pmatrix} w_t + v_{xx} - 2v^3 & 0 \\ -2v_x e^{2w} + 2v^2 e^{2w} & -w_t - v_{xx} + 2v^3 \end{pmatrix} \\ &\quad + \lambda \begin{pmatrix} -4v & -2v_x e^{-2w} - 2v^2 e^{-2w} \\ 4e^{2w} & 4v \end{pmatrix} + \lambda^2 \begin{pmatrix} 0 & -4e^{-2w} \\ 0 & 0 \end{pmatrix}. \end{aligned}$$

The new linear system, equations (6.21), recovers equations (6.18), defining w, together with the prolongation equations (6.19). The algebraic constraints correspond to the requirement that $\det \mathbf{\Psi}' = 1$ identically in λ.

Appendix A

Another generalization of the Hopf-Cole transformation

Recently Sokolov, Svinolupov and Wolf [87] introduced a technique capable of constructing generalizations of the Hopf-Cole transformation. Starting with the heat equation $v_t = v_{xx}$ they construct a contact transformation leading to the heat equation being written in the form

$$w_t = F(y, t, w_y, w_{yy}) \tag{A.1}$$

for $w(y, t)$. The new equation does not feature w explicitly. Upon differentiation with respect to y it becomes

$$w_{ty} = \frac{\partial F}{\partial y} + w_{yy}\frac{\partial F}{\partial w_y} + w_{yyy}\frac{\partial F}{\partial w_{yy}},$$

so that, following the substitutions $w_y = u$, $w_{yy} = u_y$, $w_{yt} = u_t$ and $w_{yyy} = u_{yy}$, Sokolov *et al.* obtain the differential equation

$$u_t = G(y, t, u, u_y, u_{yy}) \tag{A.2}$$

for $u(y, t)$. The heat equation and equation (A.2) are related by the generalized Hopf-Cole transformation which is given by the contact transformation followed by the relationship $u = w_y$. Of course this procedure can be repeated whenever equation (A.2) admits a contact transformation taking it into the form

$$s_t = H(z, t, s_z, s_{zz}).$$

This construction is readily interpreted in terms of the HC-projections introduced in Section 3.3. Equation (A.1) must admit the symmetry group

$$\exp(a\partial_w) : w(y,t) \mapsto w(y,t) + a, \quad a \in \mathbb{R},$$

because it does not feature w explicitly. The corresponding invariant solutions to the one-extended problem associated with equation (A.1) can be described by $w_y(y,t,w) = u(y,t)$ for suitable functions u. It is readily confirmed that u is determined by equation (A.2), so that, having found equation (A.1), the construction of Sokolov *et al.* is just a special case of that described in Section 3.3.

The contact transformation which begins the construction is also easily motivated. Given a symmetry generator $\xi(x,t,v)\partial_x + \phi(x,t,v)\partial_v$ of the heat equation, choose local coordinates in which this vector field takes the form ∂_w — this can be achieved wherever $\xi\partial_x + \phi\partial_v \neq 0$ (Proposition 1.29 of [72]). Existence of the symmetry generator ∂_w guarantees that the transformed equation takes the form of equation (A.1). Notice that the symmetry generator leaves t invariant, so that the new equation is an evolution equation with temporal variable t. The appearance of *contact* transformations here could create problems as only *point* symmetry generators $\xi\partial_x + \phi\partial_v$ are being considered. It is interesting to note that the specific examples found by Sokolov *et al.* involve only point transformations, so perhaps the generalization to contact transformations is unnecessary.

The specific equations found by Sokolov *et al.* are easily recovered using the HC-projection technique by mimicking the construction of Section 4.5 which identified the evolution equations arising as HC-projections of the PPmKdV equation. Example 3.8 listed vector fields $\{\mathbf{v}_1, \ldots, \mathbf{v}_6, \mathbf{v}_\theta\}$ spanning the symmetry algebra of the first extension of the heat equation. Restricting these vector fields to (x,t,v)-space yields the symmetry algebra for the heat equation itself. Up to conjugation by elements of the full symmetry group, the algebra spanned by $\{\mathbf{v}_1, \ldots, \mathbf{v}_6\}$ has exactly four subalgebras which generate symmetry groups leaving t invariant. These are

$$\mathrm{sp}\{\mathbf{v}_1\}, \quad \mathrm{sp}\{\mathbf{v}_3\}, \quad \mathrm{sp}\{\mathbf{v}_1, \mathbf{v}_3\}, \quad \mathrm{sp}\{\mathbf{v}_1, \mathbf{v}_3, \mathbf{v}_5\}.$$

HC-projected equations derived from the first two subalgebras were constructed in Example 3.8 as projections (e) and (f) respectively. The final two algebras lead to HC-projected equations presented in Example 4.16. As mentioned at the end of that example, these four equations comprise the entire set found by Sokolov *et al.* Unlike the approach of those authors, the role of group theory is transparent in the derivation of these equations using HC-projections.

This author believes that the method of HC-projections has certain computational advantages over the approach advocated by Sokolov *et al.* Suppose, for instance, that one wishes to construct the seven first order projections of the heat equation, given in Example 3.8, using the latter method. For each of the symmetry generators given there, one would first have to construct a coordinate change taking them into the form ∂_w. The heat equation would then have to be recalculated in terms of these new coordinates, followed by the construction of the projected equation for $u = w_y$. It seems to be much more efficient to construct a single one-extended equation and then perform the various symmetry reductions.

A more serious weakness of the method of Sokolov *et al.* involves its extension to generalized Hopf-Cole transformations of order greater than one. When G is solvable, the G-induced HC-projection of a differential equation Δ can certainly be obtained using their method. In this case, by Corollaries 4.14 and 4.15 the HC-projection can be decomposed into a sequence of first order HC-projections and the technique of Sokolov *et al.* can be applied successfully to each projection in turn. The situation is more complicated when G is not solvable. For instance, if G has Lie algebra $\mathfrak{sl}(2, \mathbb{R})$ it is impossible to decompose the HC-projection into a sequence of three first-order HC-projections. Thus, the method of Sokolov *et al.* would need to be generalized in order to handle even such simple projections as that relating the PPmKdV and KdV equations which was featured in Example 4.17.

Appendix B

A sample REDUCE session constructing partial symmetries

The partial symmetry generators of the prolongation of the mKdV equation featured in Example 5.6 are constructed using the differential geometry package EXCALC [85] which is part of the algebraic programming system REDUCE [39]. The determining equations for these symmetry generators are found by executing the following commands while running REDUCE. Begin by constructing the appropriate jet space and total derivative operators via

```
load excalc$
pform x=0,t=0,v=0,vx=0,vt=0,vxx=0,vxt=0,vtt=0,
        vxxx=0,vxxt=0,vxtt=0,vttt=0,w=0$
tvector dx,dt$
dx := @x + vx*@v + vxx*@vx + vxt*@vt
        + vxxx*@vxx + vxxt*@vxt + vxtt*@vtt + v*@w$
dt := @t + vt*@v + vxt*@vx + vtt*@vt
        + vxxt*@vxx + vxtt*@vxt + vttt*@vtt + (-vxx + 2*v**3)*@w$
```

The vector field which will become the partial symmetry generator is defined and then prolonged as far as is necessary:

```
pform f=0,g=0,h=0$
fdomain f=f(x,t,v,w),g=g(x,t,v,w),h=h(x,t,v,w)$
pform hx=0,ht=0,hxxx=0$
hx := dx|_h - vx*dx|_f - vt*dx|_g$
ht := dt|_h - vx*dt|_f - vt*dt|_g$
```

```
hxxx := dx|_(dx|_(dx|_h)) - vx*dx|_(dx|_(dx|_f)) - 3*vxx*dx|_(dx|_f)
        - 3*vxxx*dx|_f  - vt*dx|_(dx|_(dx|_g)) - 3*vxt*dx|_(dx|_g)
        - 3*vxxt*dx|_g$
```

As mentioned in Example 5.6, the determining equation is now `eqn=0` where

```
pform eqn=0$
eqn := ht + hxxx - 6*v**2*hx - 12*v*vx*h$
vt := - vxxx + 6*v**2*vx$
```

The last command restricts everything to the subvariety describing the mKdV equation.

The resulting equation is a differential equation for the functions $f(x,t,v,w)$, $g(x,t,v,w)$ and $h(x,t,v,w)$, and the appearance of higher jet variables allows one to rewrite this as a system of equations for these functions. One way to do this is via the REDUCE command

```
factor vx,vxx,vxt,vtt,vxxx,vxxt,vxtt,vttt$
```

so that the component equations can be read directly from the coefficients of the various polynomials in the jet variables. The determining equation now amounts to the following expression vanishing:

```
      4                        2      3                   3
 - (VX *(@      F + 6*@      G*V ) - VX *VXXX*@      G + VX *(3*
          V V V       V V V                  V V V

                                                       3
     @      F*V + 3*@      F + 3*@    F + 18*@      G*V  + 18*@      G*
      V V W         V V X      V W       V V W              V V X

      2             2                  2                      2
     V  + 18*@    G*V  - @      H) + 6*VX *VXX*(@    F + 3*@    G*V ) +
              V W         V V V                  V V        V V

       2                   2
    3*VX *VXT*@    G - (3*VX *VXXX)*(@      G*V + @      G + @    G) + 3
               V V                    V V W        V V X      V W
```

```
   2           2
*VX *(@     F*V  + 2*@     F*V + @     F + @   F*V + @   F + 6*
       V W W          V W X      V X X     W W       W X

            4              3              2           3
      @     G*V  + 12*@     G*V  + 6*@     G*V  + 6*@  G*V  + 6*
       V W W           V W X          V X X          W W

          2
      @   G*V  - @     H*V - @     H - @   H) - 3*VX*VXX*VXXX*
       W X         V V W       V V X     V W

                                                               3
@   G + 3*VX*VXX*(3*@   F*V + 3*@   F + @ F + 6*@   G*V  + 6*
 V V                  V W         V X     W        V W

       2
   @   G*V  - @   H) + 3*VX*VXT*(2*@   G*V + 2*@   G + @ G) + 3*
    V X        V V                 V W         V X     W

                      2
VX*VXXX*(@ F - @     G*V  - 2*@     G*V - @     G - @   G*V -
          V      V W W          V W X       V X X     W W

                                                            3
        @   G) + 3*VX*VXXT*@ G + VX*(@ F + @     F*V  + 3*
         W X                V          T     W W W

         2                        3                    2
   @     F*V  + 3*@     F*V - 4*@ F*V  + @     F - 6*@ F*V  + 6*
    W W X          W X X         W         X X X       X

       2             5              4              3
   @ G*V  + 6*@     G*V  + 18*@     G*V  + 18*@     G*V  - 24*
    T           W W W          W W X           W X X

       5             2           4             2
   @ G*V  + 6*@     G*V  - 36*@ G*V  - 3*@     H*V  - 6*@     H*V
    W           X X X          X           V W W          V W X
```

```
                                                        2
    - 3*@     H - 3*@    H*V - 3*@    H + 12*H*V) + 3*VXX *@ F + 3*
         V X X       W W         W X                        V

                                                              2
VXX*VXT*@ G - (3*VXX*VXXX)*(@    G*V + @    G) + 3*VXX*(@    F*V  +
         V                   V W      V X              W W

                                                         2
   2*@    F*V + @    F - @    H*V - @    H) + 3*VXT*(@    G*V  + 2*
      W X        X X      V W        V X              W W

                                                            3
   @    G*V + @    G) + VXXX*(3*@ F*V + 3*@ F - @ G - @      G*V  -
    W X        X X                W         X      T    W W W

              2                         3                    2
   3*@     G*V  - 3*@     G*V + 4*@ G*V   - @     G + 6*@ G*V ) +
      W W X          W X X          W        X X X       X

                                        3            2
3*VXXT*(@ G*V + @ G) - @ H - @     H*V  - 3*@     H*V  - 3*
         W       X      T    W W W           W W X

                   3                    2
@     H*V + 4*@ H*V   - @     H + 6*@ H*V )
 W X X          W        X X X       X
```

The coefficients of $v_{xx}v_{xt}$, v_xv_{xt} and v_{xxt} allow g to be simplified considerably. They imply that $g(x,t,v,w) = 3\alpha(t)$ for some function α. The commands

```
pform alpha=0$
g := 3*alpha$
fdomain alpha=alpha(t)$
```

reflect this information, reducing the determining equation to a much tidier form. The following expression must be zero:

```
      4             3
 - (VX *@     F + VX *(3*@     F*V + 3*@     F + 3*@    F - @     H) +
         V V V             V V W         V V X       V W      V V V
```

```
   2                 2           2
6*VX *VXX*@   F + 3*VX *(@     F*V  + 2*@     F*V + @     F +
           V V            V W W          V W X       V X X

 @   F*V + @   F - @    H*V - @    H - @   H) + 3*VX*VXX*(3*
  W W       W X     V V W     V V X     V W

 @   F*V + 3*@   F + @ F - @   H) + 3*VX*VXXX*@ F + VX*(@ F +
  V W          V X    W     V V                V         T

          3               2                     3
 @     F*V  + 3*@     F*V  + 3*@     F*V - 4*@ F*V  + @     F -
  W W W          W W X          W X X         W        X X X

        2                2
  6*@ F*V  - 3*@    H*V  - 6*@     H*V - 3*@     H - 3*@    H*V
     X           V W W         V W X        V X X       W W

                                  2                  2
  - 3*@    H + 18*@ ALPHA*V  + 12*H*V) + 3*VXX *@ F + 3*VXX*(
       W X          T                            V

       2
 @   F*V  + 2*@   F*V + @   F - @   H*V - @   H) + 3*VXXX*(
  W W          W X       X X     V W       V X

                                            3              2
 @ F*V + @ F - @ ALPHA) - @ H - @     H*V  - 3*@     H*V  - 3*
  W       X     T          T     W W W          W W X

                   3                     2
@     H*V + 4*@ H*V  - @     H + 6*@ H*V )
 W X X         W        X X X       X
```

The form of $f(x,t,v,w)$ can be partially determined using the coefficients of v_{xx}^2 and v_{xxx}. That is, f is independent of v and w and must satisfy

$$\frac{\partial f}{\partial x} - \frac{d\,\alpha}{d\,t} = 0,$$

so that

$$f(x,t) = x\frac{d\,\alpha}{d\,t} + \beta(t).$$

In REDUCE, this is written

```
pform beta=0$
f := x*@(alpha,t) + beta$
fdomain beta=beta(t)$
```

The determining equation has been simplified still further. It is equivalent to the following expression being identically zero:

```
  3                2
VX *@     H + 3*VX *(@      H*V + @      H + @   H) + 3*VX*VXX*@   H +
     V V V            V V W        V V X      V W                V V

                2
VX*(3*@     H*V  + 6*@      H*V + 3*@      H + 3*@    H*V + 3*@   H -
       V W W            V W X          V X X        W W          W X

                                    2
    @   ALPHA*X - 12*@ ALPHA*V  - @ BETA - 12*H*V) + 3*VXX*(@   H*V +
     T T               T          T                          V W

                                  3              2                     3
    @   H) + @ H + @      H*V  + 3*@      H*V  + 3*@      H*V - 4*@ H*V
     V X      T     W W W          W W X           W X X            W

                     2
 + @      H - 6*@ H*V
    X X X        X
```

From the coefficient of $v_x v_{xx}$ it is found that

$$\frac{\partial^2 h}{\partial v^2} = 0,$$

which implies that $h(x,t,v,w) = \gamma(x,t,w) + v\cdot\delta(x,t,w)$:

```
pform gamma=0,delta=0$
h := gamma + v*delta$
fdomain gamma=gamma(x,t,w),delta=delta(x,t,w)$
```

At this stage, the v-dependence of the functions appearing in the partial symmetry generator has been determined, so that one may factor all appearances of v in addition to the higher jet variables factored so far:

```
factor v$
```

The determining equations thus correspond to the vanishing of the following expression:

```
           2                  2
- ( - 3*VX *@ DELTA + 6*VX*V *(2*@ ALPHA - @   DELTA + 2*DELTA) + 3*
            W                      T        W W

   VX*V*( - @   GAMMA - 3*@   DELTA + 4*GAMMA) + VX*(@   ALPHA*X +
            W W           W X                        T T

     @ BETA - 3*@   GAMMA - 3*@   DELTA) - 3*VXX*V*@ DELTA - 3*VXX*
      T         W X           X X                   W

               4                                 3
   @ DELTA + V *( - @     DELTA + 4*@ DELTA) + V *( - @     GAMMA +
    X               W W W           W                 W W W

                                                     2
     4*@ GAMMA - 3*@     DELTA + 6*@ DELTA) + 3*V *( -
       W           W W X           X

     @     GAMMA + 2*@ GAMMA - @     DELTA) + V*( - 3*@     GAMMA -
      W W X          X         W X X                  W X X

      @ DELTA - @     DELTA) - (@ GAMMA + @     GAMMA))
       T        X X X            T        X X X
```

From the coefficients of v_{xx} and v_x^2, δ is a function of t only, and from the coefficient of $v^2 v_x$,

$$\delta(x,t,w) = -\frac{d\,\alpha}{d\,t}.$$

The appropriate command in REDUCE is

```
delta := - @(alpha,t)$
```

The determining equation is now almost entirely solved as it corresponds to the following expression being zero:

```
 - (3*VX*V*( - @    GAMMA + 4*GAMMA) + VX*(@    ALPHA*X + @ BETA - 3*
                W W                          T T            T

                    3                                      2
      @    GAMMA) + V *( - @     GAMMA + 4*@ GAMMA) + 3*V *( -
       W X                  W W W          W

      @     GAMMA + 2*@ GAMMA) + V*(@    ALPHA - 3*@     GAMMA) - (
       W W X            X             T T            W X X

      @ GAMMA + @      GAMMA))
       T          X X X
```

It follows from the coefficient of vv_x that

$$\gamma(x,t,w) = \phi(x,t)e^{2w} + \theta(x,t)e^{-2w}.$$

Following the commands

```
pform phi=0,theta=0$
gamma := phi*exp(2*w) + theta*exp(-2*w)$
fdomain phi=phi(x,t),theta=theta(x,t)$
```

the determining equation states that the next expression is zero:

```
        4*W            2*W                   2*W
(VX*(6*E    *@ PHI - E    *@    ALPHA*X - E    *@ BETA - 6*@ THETA) + 6*
              X              T T                  T           X

  2   4*W                           4*W            2*W
 V *(E    *@ PHI + @ THETA) + V*(6*E    *@    PHI - E    *@    ALPHA - 6*
            X        X                   X X               T T

                    4*W          4*W
   @    THETA) + E    *@ PHI + E    *@      PHI + @ THETA +
    X X                  T              X X X       T

                 2*W
 @      THETA)/E
  X X X
```

All w-dependence has been determined and the coefficients of $e^{2w}v_x$ and $e^{-2w}v_x$ can be used to show that ϕ and θ are functions of t only. Moreover, the coefficients of e^{2w} and e^{-2w} indicate that ϕ and θ are actually constant. The coefficient of v_x then implies that $\alpha(t) = c_1 + c_2 t$ and $\beta = c_3$ for some constants c_1, c_2 and c_3. Thus,

```
alpha := c1 + c2*t$
beta := c3$
phi := c4$
theta := c5$
```

so that

$$\begin{aligned} f &= c_2 x + c_3, \\ g &= 3c_1 + 3c_2 t, \\ h &= -c_2 v + c_4 e^{2w} + c_5 e^{-2w}, \end{aligned}$$

where $c_1, \ldots, c_5$ are arbitrary constants. The determining equations are satisfied and the vector space of partial symmetry generators has been identified. Since $c_1, \ldots, c_5$ are arbitrary there are five independent partial symmetry generators. They are

$$\partial_x, \quad x\partial_x + 3t\partial_t - v\partial_v, \quad \partial_t, \quad e^{2w}\partial_v, \quad e^{-2w}\partial_v,$$

the last two being *nonlocal* partial symmetry generators.

Appendix C

Derivation of some Bäcklund transformations

This appendix comprises a series of examples which construct Bäcklund transformations for several well known differential equations by applying the technique of Section 5.5 to the prolongations derived from their zero-curvature representations. They are included to provide further evidence of the widespread applicability of this method for finding Bäcklund transformations. The transformations obtained are not new, but their derivations are.

C.1 Modified KdV equation

The AKNS system [1]

$$\begin{aligned}
\begin{pmatrix} p \\ q \end{pmatrix}_x &= \begin{pmatrix} -\lambda & -u \\ -u & \lambda \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix}, \\
\begin{pmatrix} p \\ q \end{pmatrix}_t &= \begin{pmatrix} -2\lambda u^2 + 4\lambda^3 & u_{xx} - 2u^3 - 2\lambda u_x + 4\lambda^2 u \\ u_{xx} - 2u^3 + 2\lambda u_x + 4\lambda^2 u & 2\lambda u^2 - 4\lambda^3 \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix},
\end{aligned}$$

provides a zero-curvature representation for the mKdV equation

$$0 = u_t + u_{xxx} - 6u^2 u_x.$$

Straightforward calculations uncover the nonlocal partial symmetry generator $\mathbf{u} = \lambda(p^2 + q^2)\partial_u$ of the resulting prolongation of the mKdV equation. The augmented prolongation obtained by introducing a further pseudopotential r defined by

$$\begin{aligned} r_x &= -\tfrac{1}{2}\lambda p^2 - \lambda pq - \tfrac{1}{2}\lambda q^2, \\ r_t &= -\lambda(u_x + u^2 + 4\lambda u - 2\lambda^2)p^2 \\ &\quad - 2\lambda(u_x + u^2 - 6\lambda^2)pq - \lambda(u_x + u^2 - 4\lambda u - 2\lambda^2)q^2, \end{aligned}$$

admits the nonlocal symmetry generator

$$\mathbf{v}_1 = \lambda(p^2+q^2)\partial_u + r(p\partial_p + q\partial_q) - \tfrac{1}{4}(p-q)^2(p+q)(\partial_p - \partial_q) + r^2\partial_r,$$

which is equivalent to $\mathbf{u}$. The prolongation admits internal symmetry generators $\mathbf{v}_2 = p\partial_p + q\partial_q + 2r\partial_r$ and $\mathbf{v}_3 = \partial_r$, with $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ spanning a Lie algebra isomorphic to $\mathfrak{sl}(2,\mathbb{R})$. Invariants of the resulting group action are x, t and $u - 4\lambda pq/(p^2 - q^2)$ so that solutions to the corresponding HC-projected problem can be described by

$$u(x,t,p,q,r) = \frac{4\lambda pq}{p^2 - q^2} + \alpha(x,t).$$

One finds that $\alpha(x,t)$ must be a solution of the projected equation

$$0 = \alpha_t + \alpha_{xxx} - 6\alpha^2\alpha_x,$$

so that $\tilde{u} = \alpha$ satisfies the mKdV equation and an auto-Bäcklund transformation is

$$u \mapsto \tilde{u} = u - \frac{4\lambda pq}{p^2 - q^2}.$$

This transformation can be rewritten in classical form by introducing a new pseudopotential $y = q/p$. Then $y_x = 2\lambda y + (y^2 - 1)u$ and the auto-Bäcklund transformation is $\tilde{u} = u + 4\lambda y/(y^2 - 1)$. The new dependent variable $v(x,t) = \int^x u(x',t)dx'$ can be shown to satisfy the PmKdV equation

$$0 = v_t + v_{xxx} - 2v_x^3.$$

One easily verifies that

$$\frac{2y_x}{y^2-1} = 2u + \frac{4\lambda y}{y^2 - 1} = \tilde{u} + u = (\tilde{v} + v)_x,$$

so that $y = -\tanh((\tilde{v} + v)/2)$ and

$$(\tilde{v} - v)_x = \tilde{u} - u = \frac{4\lambda y}{y^2 - 1} = 2\lambda\sinh(\tilde{v} + v).$$

This recovers the spatial member of the familiar auto-Bäcklund transformation for the PmKdV equation [1].

C.2 Sine-Gordon equation

The sine-Gordon equation $u_{xt} = \sin u$ admits the zero-curvature representation

$$\begin{aligned} \begin{pmatrix} p \\ q \end{pmatrix}_x &= \begin{pmatrix} -\lambda & -\frac{1}{2}u_x \\ \frac{1}{2}u_x & \lambda \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix}, \\ \begin{pmatrix} p \\ q \end{pmatrix}_t &= \begin{pmatrix} -\frac{1}{4}\lambda^{-1}\cos u & -\frac{1}{4}\lambda^{-1}\sin u \\ -\frac{1}{4}\lambda^{-1}\sin u & \frac{1}{4}\lambda^{-1}\cos u \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix}, \end{aligned}$$

which is equivalent to the AKNS formulation [1]. The nonlocal partial symmetry generator $\mathbf{u} = (p^2+q^2)\partial_u$ leads one to augment this prolongation by introducing a new pseudopotential r defined by

$$r_x = 4\lambda pq, \quad r_t = \tfrac{1}{2}\lambda^{-1}(p^2 \sin u - 2pq\cos u - q^2 \sin u).$$

The augmented prolongation admits the genuine nonlocal symmetry generator

$$\begin{aligned} \mathbf{v}_1 &= (p^2+q^2)\partial_u + \tfrac{1}{4}r(p\partial_p + q\partial_q) \\ &\quad - \tfrac{1}{4}(p^2+q^2)(q\partial_p - p\partial_q) + \tfrac{1}{4}(r+p^2+q^2)(r-p^2-q^2)\partial_r \end{aligned}$$

which is equivalent to $\mathbf{u}$, and also features two internal symmetry generators $\mathbf{v}_2 = p\partial_p + q\partial_q + 2r\partial_r$ and $\mathbf{v}_3 = \partial_r$. Invariants of the group action determined by $\mathrm{sp}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} \cong \mathfrak{sl}(2,\mathbb{R})$ are x, t and

$$u + 2\sin^{-1}\left(\frac{p^2-q^2}{p^2+q^2}\right),$$

so that solutions to the HC-projected problem can be described by

$$u(x,t,p,q,r) = \alpha(x,t) - 2\sin^{-1}\left(\frac{p^2-q^2}{p^2+q^2}\right).$$

The resulting HC-projected equation is

$$\alpha_{xt} = -\sin\alpha,$$

whence $\tilde{u} = \alpha + \pi$ satisfies the sine-Gordon equation and an auto-Bäcklund transformation is

$$u \mapsto \tilde{u} = u + 2\sin^{-1}\left(\frac{p^2-q^2}{p^2+q^2}\right) + \pi.$$

This transformation can be written in a more familiar form following the introduction of a pseudopotential v defined by $\sin v = (p^2 - q^2)/(p^2 + q^2)$. Then $\tilde{u} = u + 2v + \pi$ and the original prolongation equations imply that $v_x = -2\lambda \cos v - u_x$. After the substitution $\lambda = -\xi/2$, one finds that

$$\begin{aligned} \tilde{u}_x &= u_x + 2(\xi \cos v - u_x) \\ &= -u_x + 2\xi \cos\left(\frac{\tilde{u} - u - \pi}{2}\right) \\ \tilde{u}_x &= -u_x + 2\xi \sin\left(\frac{\tilde{u} - u}{2}\right), \end{aligned}$$

which recovers the spatial part of the famous auto-Bäcklund transformation for the sine-Gordon equation [11]:

$$\left(\frac{\tilde{u} + u}{2}\right)_x = \xi \sin\left(\frac{\tilde{u} - u}{2}\right), \quad \left(\frac{\tilde{u} - u}{2}\right)_t = \frac{1}{\xi} \sin\left(\frac{\tilde{u} + u}{2}\right).$$

C.3 Sawada-Kotera equation

The Sawada-Kotera equation [84]

$$0 = u_t + u_{xxxxx} + 5uu_{xxx} + 5u_x u_{xx} + 5u^2 u_x$$

differs from the previous two examples in that its zero-curvature representation involves 3×3 matrices. Dodd and Fordy [18] constructed the linear system

$$\begin{aligned} \begin{pmatrix} p \\ q \\ r \end{pmatrix}_x &= \begin{pmatrix} 0 & -u & \lambda \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} p \\ q \\ r \end{pmatrix}, \\ \begin{pmatrix} p \\ q \\ r \end{pmatrix}_t &= \begin{pmatrix} A & B & \lambda u_{xx} - \lambda u^2 \\ -2u_{xx} - u^2 & C & 3\lambda u_x + 9\lambda^2 \\ -3u_x + 9\lambda & u_{xx} - u^2 & 6\lambda u \end{pmatrix} \begin{pmatrix} p \\ q \\ r \end{pmatrix}, \end{aligned}$$

where

$$\begin{aligned} A &= -u_{xxx} - uu_x - 3\lambda u, \\ B &= u_{xxxx} + 3uu_{xx} + u_x^2 + u^3 + 9\lambda^2, \\ C &= u_{xxx} + uu_x - 3\lambda u, \end{aligned}$$

which provides a zero-curvature representation. The resulting prolongation admits a nonlocal partial symmetry generator if and only if $\lambda = 0$. In that case, there is actually a genuine nonlocal symmetry generator

$$\mathbf{v}_1 = 6p\partial_u - 2(pr + q^2)\partial_p - 2qr\partial_q - r^2\partial_r$$

and two internal symmetry generators, $\mathbf{v}_2 = p\partial_p + q\partial_q + r\partial_r$ and $\mathbf{v}_3 = \partial_r$, which generate an algebra isomorphic to $\mathfrak{sl}(2, \mathbb{R})$. Invariants are x, t and $u + \frac{3}{2}p^2q^{-2}$, which lead to solutions of the three-extended problem of the form

$$u(x, t, p, q, r) = \alpha(x, t) - \tfrac{3}{2}p^2q^{-2}.$$

The undetermined function α is found to satisfy the Kaup-Kupershmidt equation [50]

$$0 = \alpha_t + \alpha_{xxxxx} - 10\alpha\alpha_{xxx} - 25\alpha_x\alpha_{xx} + 20\alpha^2\alpha_x.$$

This recovers the known Bäcklund transformation

$$u \mapsto \alpha = u + \tfrac{3}{2}p^2q^{-2}$$

relating the Sawada-Kotera and Kaup-Kupershmidt equations which was originally discovered when it was realized that both equations have Wahlquist-Estabrook prolongations leading to a common modified equation [31].

The examples featured in this appendix when viewed alongside those of Section 5.5 show the variety of auto-Bäcklund transformations which can be constructed using the nonlocal symmetries approach. Armed only with a zero-curvature representation, it has been possible to obtain auto-Bäcklund transformations for the KdV, mKdV, Harry Dym and sine-Gordon equations using infinitesimal techniques. An advantage over the various other approaches for finding auto-Bäcklund transformations is that the method advanced here has also found a Bäcklund transformation relating two different differential equations, the Sawada-Kotera and Kaup-Kupershmidt equations.

Appendix D

Verification of nonlocal symmetries of the KdV equation

The calculations which were omitted from Chapter 6 appear in this appendix. It concentrates on proving Theorem 6.5, which presented an infinite-dimensional algebra of symmetry generators of a prolongation of the KdV equation. Several other results are proven first, including the explicit form of an infinite family of nonlocal partial symmetry generators (Theorem 6.1) and two other results (Lemmas 6.3 and 6.4) which appear in the derivation of the algebra mentioned above.

D.1 Proof of Theorem 6.1

Theorem 6.1 states that the vector fields $\{\xi^{(n)}\partial_u,\ \phi^{(n)}\partial_u,\ \theta^{(n)}\partial_u : n = 1, 2, \ldots\}$ are nonlocal partial symmetry generators of a prolongation of the KdV equation. It will be shown that each $\theta^{(n)}\partial_u$ is such a symmetry generator by proving that

$$(\tilde{D}_t + \tilde{D}_x^3 + 12u\tilde{D}_x + 12u_x)(\theta^{(n)}) = 0.$$

The claimed action of the recursion operator on the characteristics of these symmetries will be verified by proving that

$$\mathcal{R}(\theta^{(n)}) = -4(1 - \delta_1^n)\theta^{(n-1)} + 4c_n u_x, \quad n = 1, 2, \ldots\ ,$$

with c_n being arbitrary constants. Almost identical calculations result from replacing $\theta^{(n)}$ by $\xi^{(n)}$ and then $\phi^{(n)}$ everywhere, which are thus omitted.

If

$$\theta^{(n)} = -2\sum_{i=0}^{n-1} p_i q_{n-1-i}$$

then

$$\begin{aligned}
\tilde{D}_x\theta^{(n)} &= 2\sum_{i=0}^{n-1}(2up_ip_{n-1-i} - q_iq_{n-1-i}) + 2\sum_{i=0}^{n-2} p_ip_{n-2-i},\\
\tilde{D}_x^2\theta^{(n)} &= 4\sum_{i=0}^{n-1}(u_xp_ip_{n-1-i} + 4up_iq_{n-1-i}) + 8\sum_{i=0}^{n-2} p_iq_{n-2-i},\\
\tilde{D}_x^3\theta^{(n)} &= 4\sum_{i=0}^{n-1}((u_{xx} - 8u^2)p_ip_{n-1-i} + 6u_xp_iq_{n-1-i} + 4uq_iq_{n-1-i})\\
&\quad - 8\sum_{i=0}^{n-2}(4up_ip_{n-2-i} - q_iq_{n-2-i}) - 8\sum_{i=0}^{n-3} p_ip_{n-3-i},
\end{aligned}$$

and

$$\begin{aligned}
\tilde{D}_t\theta^{(n)} &= -4\sum_{i=0}^{n-1}((u_{xx} + 4u^2)p_ip_{n-1-i} - 2uq_iq_{n-1-i})\\
&\quad + 8\sum_{i=0}^{n-2}(up_ip_{n-2-i} - q_iq_{n-2-i}) + 8\sum_{i=0}^{n-3} p_ip_{n-3-i}.
\end{aligned}$$

Consequently,

$$(\tilde{D}_t + \tilde{D}_x^3 + 12u\tilde{D}_x + 12u_x)(\theta^{(n)}) = 0$$

so that $\theta^{(n)}\partial_u$ is a nonlocal partial symmetry generator for all $n = 0, 1, \ldots$.

To confirm the action of $\mathcal{R}$ on $\theta^{(n)}$ notice that the general solution to the system of equations

$$\tilde{D}_x(P^{(n)}) = \theta^{(n)}, \quad \tilde{D}_t(P^{(n)}) = (-\tilde{D}_x^2 + 12u)(\theta^{(n)}),$$

is

$$P^{(n)} = -\sum_{i=0}^{n-1} p_ip_{n-1-i} + c_n,$$

with c_n an arbitrary constant. More precisely, c_n is any smooth function on the prolonged space such that $\tilde{D}_x(c_n) = \tilde{D}_t(c_n) = 0$, and can therefore be treated as a constant. Thus

$$\begin{aligned}
\mathcal{R}(\theta^{(n)}) &= (\tilde{D}_x^3 + 8u\tilde{D}_x + 4u_x)(P^{(n)})\\
&= 8\sum_{i=0}^{n-2} p_iq_{n-2-i} + 4c_nu_x\\
&= -4(1-\delta_1^n)\theta^{(n-1)} + 4c_nu_x
\end{aligned}$$

as required.

D.2 Proof of Lemma 6.3

Lemma 6.3 describes a vector field $\mathbf{u}_{-1}$ on the prolonged space and claims that it is a genuine nonlocal symmetry generator. Here it is proved that the appropriate prolongation of $\mathbf{u}_{-1}$ leaves the system of prolongation equations (6.4) and algebraic constraints $\Delta^n = 0$ invariant. The fact that there are infinitely many equations and that $\mathbf{u}_{-1}$ is a vector field on an infinite-dimensional space could create problems. Fortunately, as discussed in Section 6.1, the prolongation equations can be treated as the limit of an arbitrarily large finite system. Also, the components of $\mathbf{u}_{-1}$ are all functions of only finitely many variables so that problems with infinite-dimensionality can be avoided. Similar observations permit the constructions used to prove Lemma 6.4 and Theorem 6.5 in the following sections.

Due to the symmetry apparent in the expansion of the vector field $\mathbf{u}_{-1}$ it is sufficient to prove that the expressions

$$(\tilde{D}_x^2 + 2u)(A_i^{(1)}) + (1-\delta_i^0)A_{i-1}^{(1)} + 2p_i\xi^{(1)},$$

$$(\tilde{D}_t + 4u\tilde{D}_x - 2u_x)(A_i^{(1)}) - 4(1-\delta_i^0)\tilde{D}_x(A_{i-1}^{(1)}) - 2(p_i\tilde{D}_x - 2q_i)(\xi^{(1)}),$$

$$\sum_{i=0}^{n}\left(s_{n-i}A_i^{(1)} - r_{n-i}\tilde{D}_x(A_i^{(1)}) - q_{n-i}B_i^{(1)} + p_{n-i}\tilde{D}_x(B_i^{(1)})\right),$$

vanish on solutions to the prolongation of the KdV equation being considered here. Notice that

$$\begin{aligned}
\tilde{D}_x(A_i^{(1)}) &= -2p_0p_ir_0 - 2s_0\mathsf{D}^{-1}(p_0p_i),\\
\tilde{D}_x^2(A_i^{(1)}) &= -4p_0p_is_0 - 2p_iq_0r_0 - 2p_0q_ir_0 + 4ur_0\mathsf{D}^{-1}(p_0p_i),
\end{aligned}$$

so that

$$\begin{aligned}
&(\tilde{D}_x^2 + 2u)(A_i^{(1)}) + (1-\delta_i^0)A_{i-1}^{(1)} + 2p_i\xi^{(1)}\\
&= -2p_0q_ir_0 + 2p_iq_0r_0 - 2(1-\delta_i^0)r_0\mathsf{D}^{-1}(p_0p_{i-1})\\
&= 2\delta_i^0r_0(p_iq_0 - p_0q_i),
\end{aligned}$$

which is zero for all nonnegative integers i. Furthermore

$$\begin{aligned}
\tilde{D}_t(A_i^{(1)}) &= -4(u_xr_0 - 2us_0)\mathsf{D}^{-1}(p_0p_i)\\
&\quad - 8up_0p_ir_0 - 8q_0q_ir_0 - 8(1-\delta_i^0)p_0p_{i-1}r_0
\end{aligned}$$

and

$$\begin{aligned}(\tilde{D}_t + 4u\tilde{D}_x - 2u_x)(A_i^{(1)}) - 4(1-\delta_i^0)\tilde{D}_x(A_{i-1}^{(1)}) - 2(p_i\tilde{D}_x - 2q_i)(\xi^{(1)}) \\ = \quad 8(1-\delta_i^0)s_0\mathsf{D}^{-1}(p_0p_{i-1}) + 8p_0q_is_0 - 8p_iq_0s_0 \\ = \quad -8\delta_i^0 s_0(p_iq_0 - p_0q_i),\end{aligned}$$

which also vanishes for all nonnegative integers i. Finally,

$$\begin{aligned}&\sum_{i=0}^{n}\left(s_{n-i}A_i^{(1)} - r_{n-i}\tilde{D}_x(A_i^{(1)}) - q_{n-i}B_i^{(1)} + p_{n-i}\tilde{D}_x(B_i^{(1)})\right) \\ &= 2\sum_{i=0}^{n}((r_{n-i}s_0 - r_0s_{n-i})\mathsf{D}^{-1}(p_0p_i) + (p_0q_{n-i} - p_{n-i}q_0)\mathsf{D}^{-1}(r_0r_i)) \\ &= 2\sum_{i=0}^{n-1}(\mathsf{D}^{-1}(r_0r_{n-1-i})\mathsf{D}^{-1}(p_0p_i) - \mathsf{D}^{-1}(p_0p_{n-1-i})\mathsf{D}^{-1}(r_0r_i)) \\ &= 0\end{aligned}$$

for all nonnegative integers n. Therefore, $\mathbf{u}_{-1}$ leaves the KdV equation and those equations describing the prolongation invariant, completing the proof.

D.3 Proof of Lemma 6.4

Two more nonlocal symmetry generators, $\mathbf{v}_{-1}$ and $\mathbf{w}_{-1}$, are presented in Lemma 6.4. Recall that it is known that $\mathbf{u}_{-1}$, $\mathbf{v}_0$ and $\mathbf{w}_0$ leave the prolongation equations (6.4) and algebraic constraints $\Delta^n = 0$ invariant, so that $[\mathbf{u}_{-1}, \mathbf{v}_0]$ and $[\mathbf{w}_0, \mathbf{u}_{-1}]$ must also enjoy this property. It will be shown that $[\mathbf{u}_{-1}, \mathbf{v}_0] = -2\mathbf{v}_{-1}$, which implies that $\mathbf{v}_{-1}$ is a nonlocal symmetry generator. Similar calculations before Lemma 6.4 are sufficient to prove that $[\mathbf{w}_0, \mathbf{u}_{-1}] = -2\mathbf{w}_{-1}$ must also be a nonlocal symmetry generator.

The coefficients of ∂_u, ∂_{p_i} and ∂_{r_i} in the expansion of $[\mathbf{u}_{-1}, \mathbf{v}_0]$ are calculated explicitly. Note that

$$du([\mathbf{u}_{-1}, \mathbf{v}_0]) = -\mathbf{v}_0(\xi^{(1)}) = 4r_0s_0 = -2\phi^{(1)}$$

and, for all nonnegative integers i,

$$\begin{aligned}dp_i([\mathbf{u}_{-1}, \mathbf{v}_0]) \quad &= \quad -\mathbf{u}_{-1}(r_i) - \mathbf{v}_0(A_i^{(1)}) \\ &= \quad -B_i^{(1)} - 2r_0(p_{i+1}s_0 - p_0s_{i+1} + q_0r_{i+1} - q_{i+1}r_0) \\ &= \quad -2E_i^{(1)}\end{aligned}$$

and

$$dr_i([\mathbf{u}_{-1}, \mathbf{v}_0]) = -\mathbf{v}_0(B_i^{(1)}) = -2r_0(r_{i+1}s_0 - r_0 s_{i+1}) = -2F_i^{(1)}.$$

Because $[\mathbf{u}_{-1}, \mathbf{v}_0]$ leaves the prolonged system of equations invariant it follows that

$$dq_i([\mathbf{u}_{-1}, \mathbf{v}_0]) = \tilde{D}_x(dp_i([\mathbf{u}_{-1}, \mathbf{v}_0])) = -2\tilde{D}_x(E_i^{(1)})$$

and

$$ds_i([\mathbf{u}_{-1}, \mathbf{v}_0]) = \tilde{D}_x(dr_i([\mathbf{u}_{-1}, \mathbf{v}_0])) = -2\tilde{D}_x(F_i^{(1)}),$$

so that $[\mathbf{u}_{-1}, \mathbf{v}_0] = -2\mathbf{v}_{-1}$ takes the claimed form.

D.4 Proof of Theorem 6.5

The proof of this theorem is a long and involved process. The first task is to confirm that all of the vectors $\{\mathbf{u}_{-n}, \mathbf{v}_{-n}, \mathbf{w}_{-n} : n = 1, 2, \ldots\}$ are genuine nonlocal symmetry generators of the system. Thus they must preserve the prolongation equations as well as the algebraic constraints. It must then be shown that the Lie algebra generated by these symmetries and the internal symmetry generators introduced in Section 6.1 satisfies the given commutator relations. The Lie bracket of any two internal symmetry generators in this algebra has been given in Section 6.1 so all that remains is the verification of the commutators of internal and nonlocal symmetries as well as of the commutator of any pair of nonlocal symmetry generators. The proof is presented step by step.

Recall that all components of the nonlocal symmetry generators considered here involve only finitely many pseudopotentials, so that infinite-dimensionality will not be a problem.

Each $\mathbf{u}_{-n}$, $n = 1, 2, \ldots$ is a nonlocal symmetry generator

The proof that each $\mathbf{u}_{-n}$ preserves the prolongation of the KdV equation follows that of Lemma 6.3 very closely, which proved that $\mathbf{u}_{-1}$ is a nonlocal symmetry generator. It is sufficient to show that

$$(\tilde{D}_x^2 + 2u)(A_i^{(n)}) + (1 - \delta_i^0)A_{i-1}^{(n)} + 2p_i\xi^{(n)},$$

$$(\tilde{D}_t + 4u\tilde{D}_x - 2u_x)(A_i^{(n)}) - 4(1 - \delta_i^0)\tilde{D}_x(A_{i-1}^{(n)}) - 2(p_i\tilde{D}_x - 2q_i)(\xi^{(n)}),$$

$$\sum_{i=0}^{m} \left(s_{m-i} A_i^{(n)} - r_{m-i} \tilde{D}_x(A_i^{(n)}) - q_{m-i} B_i^{(n)} + p_{m-i} \tilde{D}_x(B_i^{(n)}) \right),$$

vanish on solutions to the prolonged system for all nonnegative integers m and positive integers n.

Since

$$A_i^{(n)} = -2 \sum_{j=0}^{n-1} r_j \mathsf{D}^{-1}(p_i p_{n-1-j}),$$

it follows that

$$\begin{aligned}
\tilde{D}_x(A_i^{(n)}) &= -2 \sum_{j=0}^{n-1} \left(p_i p_{n-1-j} r_j + s_j \mathsf{D}^{-1}(p_i p_{n-1-j})\right), \\
\tilde{D}_x^2(A_i^{(n)}) &= -2 \sum_{j=0}^{n-1} (-2u r_j \mathsf{D}^{-1}(p_i p_{n-1-j}) + 2 p_i p_{n-1-j} s_j + (p_{n-1-j} q_i + p_i q_{n-1-j}) r_j) \\
&\quad + 2 \sum_{j=0}^{n-2} r_j \mathsf{D}^{-1}(p_i p_{n-2-j}).
\end{aligned}$$

A useful identity is

$$\mathsf{D}^{-1}(p_m p_{n+1}) = \mathsf{D}^{-1}(p_{m+1} p_n) + p_{m+1} q_{n+1} - p_{n+1} q_{m+1}, \tag{D.1}$$

valid for all nonnegative integers m and n, which allows one to prove that

$$(1 - \delta_i^0) A_{i-1}^{(n)} = -2 \sum_{j=0}^{n-1} r_j (p_i q_{n-1-j} - p_{n-1-j} q_i) - 2 \sum_{j=0}^{n-2} r_j \mathsf{D}^{-1}(p_i p_{n-2-j}).$$

Consequently,

$$(\tilde{D}_x^2 + 2u)(A_i^{(n)}) + (1 - \delta_i^0) A_{i-1}^{(n)} + 2 p_i \xi^{(n)} = 0.$$

The identity

$$\tilde{D}_t(\mathsf{D}^{-1}(p_m p_n)) = 4u p_m p_n + 4 q_m q_n + 4(1 - \delta_m^0) p_{m-1} p_n + 4(1 - \delta_n^0) p_m p_{n-1},$$

valid for all nonnegative integers m and n, is also helpful as it enables one to prove that

$$\begin{aligned}
\tilde{D}_t(A_i^{(n)}) &= -4 \sum_{j=0}^{n-1} ((u_x r_j - 2u s_j) \mathsf{D}^{-1}(p_i p_{n-1-j}) + 2 r_j (u p_i p_{n-1-j} + q_i q_{n-1-j}) \\
&\qquad + 2(1 - \delta_i^0) p_{i-1} p_{n-1-j} r_j) \\
&\quad - 8 \sum_{j=0}^{n-2} (p_i p_{n-2-j} r_j + s_j \mathsf{D}^{-1}(p_i p_{n-2-j})).
\end{aligned}$$

Then

$$
\begin{aligned}
&(\tilde{D}_t + 4u\tilde{D}_x - 2u_x)(A_i^{(n)}) - 4(1-\delta_i^0)\tilde{D}_x(A_{i-1}^{(n)}) - 2(p_i\tilde{D}_x - 2q_i)(\xi^{(n)}) \\
&= 8\sum_{j=0}^{n-2} s_j((1-\delta_i^0)\mathsf{D}^{-1}(p_{i-1}p_{n-1-j}) - \mathsf{D}^{-1}(p_ip_{n-2-j}) + p_{n-1-j}q_i - p_iq_{n-1-j}) \\
&\quad + 8s_{n-1}((1-\delta_i^0)\mathsf{D}^{-1}(p_0p_{i-1}) + p_0q_i - p_iq_0),
\end{aligned}
$$

which vanishes for all nonnegative integers i and positive integers n by the definition of $\mathsf{D}^{-1}(p_0p_{i-1})$ and equations (D.1).

Finally, it must be confirmed that the algebraic constraints are preserved. For each $m = 0, 1, \dots$ and $n = 1, 2, \dots$ let

$$
\begin{aligned}
\#_n^m &= \sum_{i=0}^{m}\left(s_{m-i}A_i^{(n)} - r_{m-i}\tilde{D}_x(A_i^{(n)}) - q_{m-i}B_i^{(n)} + p_{m-i}\tilde{D}_x(B_i^{(n)})\right) \\
&= 2\sum_{i=0}^{m}\sum_{j=0}^{n-1}((r_{m-i}s_j - r_js_{m-i})\mathsf{D}^{-1}(p_ip_{n-1-j}) \\
&\qquad\qquad + (p_jq_{m-i} - p_{m-i}q_j)\mathsf{D}^{-1}(r_ir_{n-1-j})).
\end{aligned}
$$

The algebraic constraints are preserved if and only if $\#_n^m$ vanishes for all appropriate values of m and n. Equation (D.1) implies that

$$
\begin{aligned}
\#_{n+1}^m &= 2\sum_{i=0}^{m}\sum_{j=0}^{n-1}((r_{m-i}s_j - r_js_{m-i})\mathsf{D}^{-1}(p_{i+1}p_{n-1-j}) \\
&\qquad\qquad + (p_jq_{m-i} - p_{m-i}q_j)\mathsf{D}^{-1}(r_{i+1}r_{n-1-j})) \\
&\quad + 2\sum_{j=0}^{n}\sum_{i=0}^{m}(p_js_{n-j}(q_{m-i}r_{i+1} - q_{i+1}r_{m-i}) + q_jr_{n-j}(p_{m-i}s_{i+1} - p_{i+1}s_{m-i}) \\
&\qquad\qquad + q_js_{n-j}(p_{i+1}r_{m-i} - p_{m-i}r_{i+1}) + p_jr_{n-j}(q_{i+1}s_{m-i} - q_{m-i}s_{i+1})) \\
&= 2\sum_{i=0}^{m}\sum_{j=0}^{n-1}((r_{m-i}s_j - r_js_{m-i})\mathsf{D}^{-1}(p_{i+1}p_{n-1-j}) \\
&\qquad\qquad + (p_jq_{m-i} - p_{m-i}q_j)\mathsf{D}^{-1}(r_{i+1}r_{n-1-j})) \\
&\quad + 2\sum_{j=0}^{n-1}((r_{m+1}s_j - r_js_{m+1})(p_{n-j}q_0 - p_0q_{n-j}) \\
&\qquad\qquad + (p_jq_{m+1} - p_{m+1}q_j)(r_{n-j}s_0 - r_0s_{n-j})) \\
&= 2\sum_{i=-1}^{m}\sum_{j=0}^{n-1}((r_{m-i}s_j - r_js_{m-i})\mathsf{D}^{-1}(p_{i+1}p_{n-1-j}) \\
&\qquad\qquad + (p_jq_{m-i} - p_{m-i}q_j)\mathsf{D}^{-1}(r_{i+1}r_{n-1-j})) \\
&= \#_n^{m+1}.
\end{aligned}
$$

From the proof of Lemma 6.3, $\#_1^m = 0$ for all nonnegative integers m. Furthermore, if $\#_n^m = 0$ for all nonnegative integers m and some $n \in \{1, 2, \ldots\}$ then $\#_{n+1}^m = \#_n^{m+1} = 0$ for all nonnegative integers m. By induction on n, $\#_n^m = 0$ for all nonnegative integers m and positive integers n and the proof of this step is complete.

Each $\mathbf{v}_{-n}$ and $\mathbf{w}_{-n}$, $n = 1, 2, \ldots$, is a nonlocal symmetry generator

Verification of the other nonlocal symmetry generators is now straightforward. The fact that each $\mathbf{u}_{-n}$ is a symmetry generator, as are the vector fields $\mathbf{v}_0$ and $\mathbf{w}_0$, means that $[\mathbf{v}_0, \mathbf{u}_{-n}]$ and $[\mathbf{w}_0, \mathbf{u}_{-n}]$ must also be symmetries of the prolonged system. These new vectors will be calculated explicitly and be shown to equal $2\mathbf{v}_{-n}$ and $-2\mathbf{w}_{-n}$ respectively.

Notice that

$$\mathbf{v}_0(\xi^{(n)}) = 2\phi^{(n)}, \quad n = 1, 2, \ldots ,$$

and that for all nonnegative integers m and n

$$\begin{aligned} \mathbf{v}_0(\mathsf{D}^{-1}(p_m p_n)) &= -\mathsf{D}^{-1}(p_m r_n + p_n r_m), \\ \mathbf{v}_0(\mathsf{D}^{-1}(r_m r_n)) &= 0. \end{aligned} \tag{D.2}$$

Calculation of expressions such as $\mathbf{v}_0(A_i^{(n)})$ can now be performed easily, enabling one to evaluate the Lie bracket of $\mathbf{v}_0$ and any of the vector fields introduced in Theorem 6.5. For instance,

$$\begin{aligned} du([\mathbf{v}_0, \mathbf{u}_{-n}]) &= \mathbf{v}_0(\xi^{(n)}) &&= 2\phi^{(n)}, \\ dp_i([\mathbf{v}_0, \mathbf{u}_{-n}]) &= \mathbf{v}_0(A_i^{(n)}) + B_i^{(n)} &&= 2E_i^{(n)}, \\ dr_i([\mathbf{v}_0, \mathbf{u}_{-n}]) &= \mathbf{v}_0(B_i^{(n)}) &&= 2F_i^{(n)}, \end{aligned}$$

and, because $[\mathbf{v}_0, \mathbf{u}_{-n}]$ must preserve the prolongation equations, it follows that

$$dq_i([\mathbf{v}_0, \mathbf{u}_{-n}]) = 2\tilde{D}_x(E_i^{(n)}), \quad ds_i([\mathbf{v}_0, \mathbf{u}_{-n}]) = 2\tilde{D}_x(F_i^{(n)}).$$

Thus $[\mathbf{v}_0, \mathbf{u}_{-n}] = 2\mathbf{v}_{-n}$, so that for all $n = 1, 2, \ldots$ the vector field $\mathbf{v}_{-n}$ given in Theorem 6.5 is a nonlocal symmetry generator. Similar calculations confirm that for all $n = 1, 2, \ldots$

$$\mathbf{w}_0(\xi^{(n)}) = -2\theta^{(n)},$$

and for all nonnegative integers m and n

$$\begin{aligned} \mathbf{w}_0(\mathsf{D}^{-1}(p_m p_n)) &= 0, \\ \mathbf{w}_0(\mathsf{D}^{-1}(r_m r_n)) &= \mathsf{D}^{-1}(p_m r_n + p_n r_m). \end{aligned} \tag{D.3}$$

By evaluating various coefficients in turn, one easily finds that $[\mathbf{w}_0, \mathbf{u}_{-n}] = -2\mathbf{w}_{-n}$, so that each vector field $\mathbf{w}_{-n}$ introduced in Theorem 6.5 is also a nonlocal symmetry generator.

The remainder of this appendix is devoted to uncovering the algebraic structure of the algebra generated by the internal and nonlocal symmetry generators introduced thus far. It will be shown that the table of commutators

	$\mathbf{u}_n$	$\mathbf{v}_n$	$\mathbf{w}_n$
$\mathbf{u}_m$	0	$-2\mathbf{v}_{m+n}$	$2\mathbf{w}_{m+n}$
$\mathbf{v}_m$	$2\mathbf{v}_{m+n}$	0	$\mathbf{u}_{m+n}$
$\mathbf{w}_m$	$-2\mathbf{w}_{m+n}$	$-\mathbf{u}_{m+n}$	0

is valid for all integers m and n. This formidable task is split into several steps. Notice that, from the results of Section 6.1, the table certainly holds when m and n are nonnegative.

Commutators when $m = 0$ and $n \in \mathbb{Z}$

It will only be necessary to consider strictly negative values of n since the commutators of pairs of internal symmetries are already known. Furthermore, it has been shown above that

$$[\mathbf{v}_0, \mathbf{u}_{-n}] = 2\mathbf{v}_{-n}, \quad [\mathbf{w}_0, \mathbf{u}_{-n}] = -2\mathbf{w}_{-n}, \quad n = 1, 2, \dots ,$$

so that only seven commutators remain, three of which will be calculated explicitly. It is easily checked that

$$\mathbf{v}_0(\phi^{(n)}) = 0, \quad \mathbf{v}_0(\theta^{(n)}) = \xi^{(n)}, \quad \mathbf{w}_0(\theta^{(n)}) = 0, \quad n = 1, 2, \dots ,$$

and that for all nonnegative integers m and n

$$\begin{aligned} \mathbf{v}_0(\mathsf{D}^{-1}(p_m r_n + p_n r_m)) &= -2\mathsf{D}^{-1}(r_m r_n), \\ \mathbf{w}_0(\mathsf{D}^{-1}(p_m r_n + p_n r_m)) &= 2\mathsf{D}^{-1}(p_m p_n). \end{aligned}$$

Equations (D.2) and (D.3) combine with these relationships to show that

$$[\mathbf{v}_0, \mathbf{v}_{-n}] = 0, \quad [\mathbf{v}_0, \mathbf{w}_{-n}] = \mathbf{u}_{-n}, \quad [\mathbf{w}_0, \mathbf{w}_{-n}] = 0.$$

Use can now be made of the Jacobi identity to prove that

$$\begin{aligned}
[\mathbf{w}_{-n}, \mathbf{u}_0] &= [\mathbf{w}_{-n}, [\mathbf{v}_0, \mathbf{w}_0]] = -[\mathbf{u}_{-n}, \mathbf{w}_0] = -2\mathbf{w}_{-n}, \\
[\mathbf{u}_0, \mathbf{u}_{-n}] &= [\mathbf{u}_0, [\mathbf{v}_0, \mathbf{w}_{-n}]] = -2[\mathbf{v}_0, \mathbf{w}_{-n}] + 2[\mathbf{v}_0, \mathbf{w}_{-n}] = 0, \\
[\mathbf{u}_0, \mathbf{v}_{-n}] &= \tfrac{1}{2}[\mathbf{u}_0, [\mathbf{v}_0, \mathbf{u}_{-n}]] = -[\mathbf{v}_0, \mathbf{u}_{-n}] = -2\mathbf{v}_{-n}, \\
[\mathbf{w}_0, \mathbf{v}_{-n}] &= \tfrac{1}{2}[\mathbf{w}_0, [\mathbf{v}_0, \mathbf{u}_{-n}]] = -\tfrac{1}{2}[\mathbf{u}_0, \mathbf{u}_{-n}] - [\mathbf{v}_0, \mathbf{w}_{-n}] = -\mathbf{u}_{-n},
\end{aligned}$$

confirming the structure of the commutator table in the special case $m = 0$.

Commutators when $m \in \{0, 1\}$ and $n \in \mathbb{Z}$

The range of validity of the commutator table is extended to $m \in \{0, 1\}$ by explicitly computing $[\mathbf{v}_1, \mathbf{w}_{-n}]$ and then appealing to the Jacobi identity. It is easily shown that

$$\mathbf{v}_1(\theta^{(n)}) = (1 - \delta_1^n)\xi^{(n-1)}, \quad n = 1, 2, \ldots,$$

and therefore

$$du([\mathbf{v}_1, \mathbf{w}_{-n}]) = (1 - \delta_1^n)\xi^{(n-1)}, \quad n = 1, 2, \ldots.$$

The identities

$$\begin{aligned}
\mathbf{v}_1(\mathsf{D}^{-1}(p_m r_n + p_n r_m)) &= -2(1 - \delta_m^0)\mathsf{D}^{-1}(r_{m-1} r_n) + r_m s_n - r_n s_m, \\
\mathbf{v}_1(\mathsf{D}^{-1}(p_m p_n)) &= -(1 - \delta_m^0)\mathsf{D}^{-1}(p_{m-1} r_n + p_n r_{m-1}) + p_m s_n - q_m r_n,
\end{aligned}$$

are valid for all nonnegative integers m and n and are particularly useful. They simplify the construction of the remaining components of $[\mathbf{v}_1, \mathbf{w}_{-n}]$ considerably. For example,

$$\begin{aligned}
\mathbf{v}_1(H_i^{(n)}) &= (1 - \delta_1^n)B_i^{(n-1)} - r_i \sum_{j=0}^{n-1}(p_j s_{n-1-j} - q_{n-1-j} r_j) \\
&= (1 - \delta_1^n)B_i^{(n-1)} - \delta_1^n r_i,
\end{aligned}$$

where the algebraic constraint $\Delta^{n-1} = 0$ has been used to simplify the last term. It follows that

$$dr_i([\mathbf{v}_1, \mathbf{w}_{-n}]) = \mathbf{v}_1(H_i^{(n-1)}) = (1 - \delta_1^n)B_i^{(n-1)} - \delta_1^n r_i.$$

The calculation of the coefficient of ∂_{p_i} is slightly more complicated. One finds that

$$\begin{aligned} \mathrm{v}_1(G_i^{(n)}) &= -(1-\delta_i^0)\sum_{j=0}^{n-1} p_j \mathsf{D}^{-1}(p_{i-1}r_{n-1-j}+p_{n-1-j}r_{i-1}) \\ &\quad + \delta_1^n p_i + \sum_{j=0}^{n-1} r_j(p_i q_{n-1-j} - p_{n-1-j}q_i) - \sum_{j=0}^{n-2} r_j \mathsf{D}^{-1}(p_i p_{n-2-j}), \end{aligned}$$

where $\Delta^{n-1} = 0$ and the pair of identities above have again been used. Equation (D.1) implies that

$$p_i q_{n-1-j} - p_{n-1-j}q_i = (1-\delta_i^0)\mathsf{D}^{-1}(p_{i-1}p_{n-1-j}) - (1-\delta_j^{n-1})\mathsf{D}^{-1}(p_i p_{n-2-j}),$$

enabling one to show that

$$\mathrm{v}_1(G_i^{(n)}) = (1-\delta_1^n)A_i^{(n-1)} + \delta_1^n p_i - (1-\delta_i^0)H_{i-1}^{(n)}.$$

Therefore,

$$dp_i([\mathrm{v}_1,\mathbf{w}_{-n}]) = \mathrm{v}_1(G_i^{(n)}) + (1-\delta_i^0)H_{i-1}^{(n)} = (1-\delta_1^n)A_i^{(n-1)} + \delta_1^n p_i,$$

and, following the usual procedure to find the coefficients of ∂_{q_i} and ∂_{s_i}, it follows that

$$[\mathrm{v}_1,\mathbf{w}_{-n}] = (1-\delta_1^n)\mathbf{u}_{-(n-1)} + \delta_1^n\mathbf{u}_0 = \mathbf{u}_{1-n}, \quad n = 1,2,\dots\ .$$

Completion of this step follows quickly using the Jacobi identity. One finds that

$$[\mathrm{v}_1,\mathbf{u}_{-n}] = [\mathrm{v}_1,[\mathrm{v}_0,\mathbf{w}_{-n}]] = [\mathrm{v}_0,\mathbf{u}_{1-n}] = 2\mathrm{v}_{1-n},$$

$$[\mathbf{u}_1,\mathbf{w}_{-n}] = [\mathbf{w}_{-n},[\mathbf{w}_0,\mathrm{v}_1]] = [\mathbf{w}_0,-\mathbf{u}_{1-n}] = 2\mathbf{w}_{1-n},$$

$$[\mathrm{v}_1,\mathrm{v}_{-n}] = [\mathrm{v}_1,\tfrac{1}{2}[\mathrm{v}_0,\mathbf{u}_{-n}]] = \tfrac{1}{2}[\mathrm{v}_0,2\mathrm{v}_{1-n}] = 0,$$

$$[\mathbf{u}_1,\mathbf{u}_{-n}] = [\mathbf{u}_{-n},[\mathbf{w}_0,\mathrm{v}_1]] = [2\mathbf{w}_{-n},\mathrm{v}_1] + [\mathbf{w}_0,-2\mathrm{v}_{1-n}] = 0,$$

$$[\mathbf{w}_1,\mathbf{w}_{-n}] = [\mathbf{w}_{-n},\tfrac{1}{2}[\mathbf{w}_0,\mathbf{u}_1]] = \tfrac{1}{2}[\mathbf{w}_0,-2\mathbf{w}_{1-n}] = 0,$$

$$[\mathbf{u}_1,\mathrm{v}_{-n}] = [\mathrm{v}_{-n},[\mathbf{w}_0,\mathrm{v}_1]] = [\mathbf{u}_{-n},\mathrm{v}_1] = -2\mathrm{v}_{1-n},$$

$$[\mathbf{w}_1,\mathbf{u}_{-n}] = [\mathbf{u}_{-n},\tfrac{1}{2}[\mathbf{w}_0,\mathbf{u}_1]] = \tfrac{1}{2}[2\mathbf{w}_{-n},\mathbf{u}_1] = -2\mathbf{w}_{1-n},$$

$$[\mathbf{w}_1,\mathrm{v}_{-n}] = [\mathrm{v}_{-n},\tfrac{1}{2}[\mathbf{w}_0,\mathbf{u}_1]] = \tfrac{1}{2}[\mathbf{u}_{-n},\mathbf{u}_1] + \tfrac{1}{2}[\mathbf{w}_0,2\mathrm{v}_{1-n}] = -\mathbf{u}_{1-n}.$$

Thus, the commutator table is confirmed for $m \in \{0,1\}$ and $n \in \mathbb{Z}$, after using the results of Section 6.1 to extend the range of n to incorporate internal symmetries.

Commutators when $m \in \{0, 1, \ldots\}$ and $n \in \mathbb{Z}$

A straightforward induction proof extends the validity of the commutator table to all nonnegative values of m. Suppose that the commutator table is valid for all $m \in \{0, \ldots, r\}$ and $n \in \mathbb{Z}$ for some positive integer r. Let c_1, c_2, c_3 and n_1, n_2 and n_3 be arbitrary real numbers and integers, respectively, and set

$$\mathbf{x} = c_1 \mathbf{u}_{n_1} + c_2 \mathbf{v}_{n_2} + c_3 \mathbf{w}_{n_3}.$$

Then

$$\begin{aligned}
[\mathbf{x}, \mathbf{u}_{r+1}] &= [\mathbf{x}, [\mathbf{v}_r, \mathbf{w}_1]] \\
&= [c_1[\mathbf{u}_{n_1}, \mathbf{v}_r] + c_2[\mathbf{v}_{n_2}, \mathbf{v}_r] + c_3[\mathbf{w}_{n_3}, \mathbf{v}_r], \mathbf{w}_1] \\
&\quad + [\mathbf{v}_r, c_1[\mathbf{u}_{n_1}, \mathbf{w}_1] + c_2[\mathbf{v}_{n_2}, \mathbf{w}_1] + c_3[\mathbf{w}_{n_3}, \mathbf{w}_1]] \\
&= -2c_1[\mathbf{v}_{n_1+r}, \mathbf{w}_1] - c_3[\mathbf{u}_{n_3+r}, \mathbf{w}_1] + 2c_1[\mathbf{v}_r, \mathbf{w}_{n_1+1}] + c_2[\mathbf{v}_r, \mathbf{u}_{n_2+1}] \\
&= 2c_2 \mathbf{v}_{n_2+r+1} - 2c_3 \mathbf{w}_{n_3+r+1}
\end{aligned}$$

and equating the coefficients of c_k with those in

$$[\mathbf{x}, \mathbf{u}_{r+1}] = c_1[\mathbf{u}_{n_1}, \mathbf{u}_{r+1}] + c_2[\mathbf{v}_{n_2}, \mathbf{u}_{r+1}] + c_3[\mathbf{w}_{n_3}, \mathbf{u}_{r+1}]$$

shows that

$$[\mathbf{u}_{r+1}, \mathbf{u}_n] = 0, \quad [\mathbf{u}_{r+1}, \mathbf{v}_n] = -2\mathbf{v}_{n+r+1}, \quad [\mathbf{u}_{r+1}, \mathbf{w}_n] = 2\mathbf{w}_{n+r+1},$$

for all integers n. Repeating these calculations with $\mathbf{u}_{r+1}$ replaced by $\mathbf{v}_{r+1} = \frac{1}{2}[\mathbf{v}_1, \mathbf{u}_r]$ and $\mathbf{w}_{r+1} = \frac{1}{2}[\mathbf{u}_1, \mathbf{w}_r]$ confirms that the commutator table is correct when $m = r + 1$. By induction on r the commutator table is valid for all $m = 0, 1, \ldots$ and $n \in \mathbb{Z}$.

Evaluate $[\mathbf{v}_{-1}, \mathbf{w}_{-n}]$ for all $n = 1, 2, \ldots$

The final steps involve evaluating commutators of pairs of nonlocal symmetry generators. Only $[\mathbf{v}_{-1}, \mathbf{w}_{-n}]$ for all $n = 1, 2, \ldots$ needs to be computed explicitly with the Jacobi identity then yielding all other commutators. Notice that

$$du([\mathbf{v}_{-1}, \mathbf{w}_{-n}]) \;=\; \mathbf{v}_{-1}(\theta^{(n)}) - \mathbf{w}_{-n}(\phi^{(1)})$$

$$
\begin{aligned}
&= 2\sum_{j=0}^{n-1}((p_0q_j+p_jq_0)\mathsf{D}^{-1}(r_0r_{n-1-j}) \\
&\qquad -(r_0s_j+r_js_0)\mathsf{D}^{-1}(p_0p_{n-1-j})) \\
&= 2\sum_{j=0}^{n-1}(p_0q_jr_{n-j}s_0-p_jq_0r_0s_{n-j}+p_jq_0r_{n-j}s_0-p_0q_jr_0s_{n-j}) \\
&\quad +2\sum_{j=1}^{n}(p_0q_jr_{n-j}s_0-p_jq_0r_0s_{n-j}-p_jq_0r_{n-j}s_0+p_0q_jr_0s_{n-j}) \\
&= 4\sum_{j=0}^{n}(p_0q_jr_{n-j}s_0-p_jq_0r_0s_{n-j}) \\
&= \xi^{(n+1)},
\end{aligned}
$$

where the algebraic constraints $\Delta^0=\Delta^n=0$ have been used in the final step.

It should be possible to calculate the coefficients of ∂_{p_i} and ∂_{r_i}, as has been done with other commutators, but the intervening calculations become too complicated to handle in this case. Instead, another approach will be followed to prove that $[\mathbf{v}_{-1},\mathbf{w}_{-n}]=\mathbf{u}_{-(n+1)}$. At this stage, since $du([\mathbf{v}_{-1},\mathbf{w}_{-n}])=du(\mathbf{u}_{-(n+1)})$ it follows that $[\mathbf{v}_{-1},\mathbf{w}_{-n}]=\mathbf{u}_{-(n+1)}+\mathbf{x}_{-n}$ for all $n=1,2,\dots$, where each $\mathbf{x}_{-n}$ must be an internal symmetry generator of the prolongation of the KdV equation being considered. It will be shown that each $\mathbf{x}_{-n}$ must be zero. A crucial observation is that, due to the form of $\mathbf{u}_{-(n+1)}$, $\mathbf{v}_{-1}$ and $\mathbf{w}_{-n}$, the components of $\mathbf{x}_{-n}$ must be polynomials of finitely many of the pseudopotentials. The following lemma provides useful information about these vector fields.

Lemma D.1 *Let $\Delta[u]=0$ denote a system of n-th order differential equations with Wahlquist-Estabrook prolongation described by*

$$y_i^a=\sum_{\mu=1}^{s}\sigma_i^\mu(x,u^{(n-1)})X_\mu^a(y),\quad i=1,\dots,p,\ \ a=1,\dots,r,$$

where the set of vector-valued functions $\{(\sigma_1^\mu,\dots,\sigma_p^\mu):\mu=1,\dots,s\}$ is linearly independent over $\mathbb{R}$. If $\mathbf{v}=\sum_{a=1}^r\phi^a(y)\partial_{y^a}$ is an internal symmetry generator of this prolongation then $[\mathbf{x}_\mu,\mathbf{v}]=0$ for all $\mu=1,\dots,s$, where $\mathbf{x}_\mu=\sum_{a=1}^r X_\mu^a(y)\partial_{y^a}$.

PROOF: $\mathbf{v}$ is an internal symmetry generator if and only if

$$\tilde{D}_{x^i}(\phi^a)=\mathbf{v}\left(\sum_{\mu=1}^{s}\sigma_i^\mu X_\mu^a\right)=\sum_{\mu=1}^{s}\sigma_i^\mu\left(\sum_{b=1}^{r}\phi^b\frac{\partial X_\mu^a}{\partial y^b}\right),\quad i=1,\dots,p,\ \ a=1,\dots,r,$$

on solutions to the prolonged system. But

$$\tilde{D}_{x^i}(\phi^a) = \sum_{b=1}^{r}\left(\sum_{\mu=1}^{s}\sigma_i^\mu X_\mu^b\right)\frac{\partial \phi^a}{\partial y^b}, \quad i=1,\ldots,p, \quad a=1,\ldots,r,$$

because of the assumed form of $\mathbf{v}$. That is,

$$0 = \sum_{\mu=1}^{s}\sigma_i^\mu[\mathbf{x}_\mu, \mathbf{v}], \quad i=1,\ldots,p,$$

and the linear independence of $\{(\sigma_1^\mu,\ldots,\sigma_p^\mu) : \mu = 1,\ldots,s\}$ proves that $[\mathbf{x}_\mu, \mathbf{v}] = 0$ for every $\mu = 1,\ldots,s$. $\square$

As a consequence of this lemma, $\mathbf{x}_{-n}$ must commute with all vector fields in $\mathfrak{g}_L$ which were described in Section 6.1. One would like to identify $\mathfrak{g}_L$ with the Lie algebra of left-invariant vector fields on some Lie group and then use Lemma 4.9 to show that $\mathbf{x}_{-n}$ must be right-invariant. Unfortunately, the infinite-dimensionality of $\mathfrak{g}_L$ might create problems so another way must be found to prove that $\mathbf{x}_{-n} \in \mathfrak{g}_R$.

Suppose the local coordinate expansion of $\mathbf{x}_{-n}$ is

$$\mathbf{x}_{-n} = \sum_{i=0}^{\infty}\left(\zeta_n^i\partial_{p_i} + \eta_n^i\partial_{q_i} + \rho_n^i\partial_{r_i} + \sigma_n^i\partial_{s_i}\right).$$

By the earlier discussion there exists a sequence of integers $\{N_i^n : i = 0,1,\ldots\}$ such that ζ_n^i is a function of $(p_j, q_j, r_j, s_j : j = 0,\ldots,N_i^n)$, only. It will be shown that one can assume that $N_i^n \le i$ for all i and n. By the preceding lemma, if $N_i^n > i$ then

$$0 = dp_i([\mathbf{w}^*_{N_i^n}, \mathbf{x}_{-n}]) = \mathbf{w}^*_{N_i^n}(\zeta_n^i) = q_0\frac{\partial\zeta_n^i}{\partial p_{N_i^n}} + s_0\frac{\partial\zeta_n^i}{\partial r_{N_i^n}}.$$

Using $\mathbf{v}^*_{N_i^n}$ and $\mathbf{u}^*_{N_i^n}$ in place of $\mathbf{w}^*_{N_i^n}$ one also finds that

$$\begin{aligned} 0 &= -p_0\frac{\partial\zeta_n^i}{\partial q_{N_i^n}} - r_0\frac{\partial\zeta_n^i}{\partial s_{N_i^n}}, \\ 0 &= p_0\frac{\partial\zeta_n^i}{\partial p_{N_i^n}} - q_0\frac{\partial\zeta_n^i}{\partial q_{N_i^n}} + r_0\frac{\partial\zeta_n^i}{\partial r_{N_i^n}} - s_0\frac{\partial\zeta_n^i}{\partial s_{N_i^n}}, \end{aligned}$$

so that ζ_n^i must be a function of $(p_j, q_j, r_j, s_j : j = 0,\ldots,N_i^n - 1)$ and the variable

$$p_0 s_{N_i^n} + p_{N_i^n}s_0 - q_0 r_{N_i^n} - q_{N_i^n}r_0$$

only. However, using the algebraic constraint $\Delta^{N_i^n} = 0$, the last expression can be replaced by a polynomial involving just $(p_j, q_j, r_j, s_j : j = 0,\ldots,N_i^n - 1)$. Therefore, whenever N_i^n is greater than i it can be replaced by $N_i^n - 1$, leading one to the conclusion that ζ_n^i depends on at most $(p_j, q_j, r_j, s_j : j = 0, \ldots, i)$. Similar calculations yield identical dependencies for η_n^i, ρ_n^i and σ_n^i.

Consequently, $\mathbf{x}_{-n} = \lim_{N\to\infty} \mathbf{x}_{-n}^{(N)}$ where each

$$\mathbf{x}_{-n}^{(N)} = \sum_{i=0}^{N} \left(\zeta_n^i \partial_{p_i} + \eta_n^i \partial_{q_i} + \rho_n^i \partial_{r_i} + \sigma_n^i \partial_{s_i}\right)$$

is a vector field on $\mathbb{R}^{4(n+1)}$. Because $\mathbf{x}_{-n}$ is an internal symmetry generator of the infinite-dimensional prolongation of the KdV equation, each $\mathbf{x}_{-n}^{(N)}$ must be an internal symmetry generator of the prolongation with pseudopotential space $G^{(N)}$ which was introduced in Section 6.1. Lemma D.1 implies that $\mathbf{x}_{-n}^{(N)}$ must commute with all vectors in $\mathfrak{g}_L^{(N)}$. By Lemma 4.9, when restricted to the connected component of $G^{(N)}$ containing the identity, $\mathbf{x}_{-n}^{(N)}$ must be right-invariant. Thus there exist constants $\{a_n^i, b_n^i, c_n^i : i = 0, 1, \ldots\}$ such that

$$\mathbf{x}_{-n}^{(N)} = \sum_{i=0}^{N} \left(a_n^i \mathbf{u}_i^{(N)} + b_n^i \mathbf{v}_i^{(N)} + c_n^i \mathbf{w}_i^{(N)}\right) + \mathbf{y}_{-n}^{(N)},$$

where $\mathbf{y}_{-n}^{(N)}$ is a vector field which commutes with $\mathfrak{g}_L^{(N)}$ and which vanishes on the connected component of $G^{(N)}$ containing the identity. The observation that the components of $\mathbf{y}_{-n}^{(N)}$ are polynomials on $G^{(N)}$ shows that $\mathbf{y}_{-n}^{(N)} = 0$. Taking $N \to \infty$ it follows that

$$\mathbf{x}_{-n} = \sum_{i=0}^{\infty} \left(a_n^i \mathbf{u}_i + b_n^i \mathbf{v}_i + c_n^i \mathbf{w}_i\right) \tag{D.4}$$

and $\mathbf{x}_{-n} \in \mathfrak{g}_R$.

The proof of this step is almost complete because, from that part of the commutator table which is known to be correct, for all positive integers n

$$[\mathbf{u}_n, \mathbf{x}_{-n}] = [\mathbf{u}_n, [\mathbf{v}_{-1}, \mathbf{w}_{-n}] - \mathbf{u}_{-(n+1)}] = -2[\mathbf{v}_{n-1}, \mathbf{w}_{-n}] + 2[\mathbf{v}_{-1}, \mathbf{w}_0] = 0,$$

$$[\mathbf{v}_n, \mathbf{x}_{-n}] = [\mathbf{v}_n, [\mathbf{v}_{-1}, \mathbf{w}_{-n}] - \mathbf{u}_{-(n+1)}] = [\mathbf{v}_{-1}, \mathbf{u}_0] - 2\mathbf{v}_{-1} = 0.$$

By comparing these results with equation (D.4), it is easy to prove that $\mathbf{x}_{-n}=0$ for all $n = 1, 2, \ldots$. That is, $[\mathbf{v}_{-1}, \mathbf{w}_{-n}] = \mathbf{u}_{-(n+1)}$.

Commutators when $m, n \in \mathbb{Z}$

Completing the proof of Theorem 6.5 is now straightforward. The range of validity of the commutator table is readily extended to all values of $m \in \{-1, 0, \ldots\}$ by judicious application of the Jacobi identity. For all integers n,

$$[\mathbf{v}_{-1}, \mathbf{u}_{-n}] = [\mathbf{v}_{-1}, [\mathbf{v}_0, \mathbf{w}_{-n}]] = [\mathbf{v}_0, \mathbf{u}_{-(n+1)}] = 2\mathbf{v}_{-(n+1)},$$

$$[\mathbf{u}_{-1}, \mathbf{w}_{-n}] = [\mathbf{w}_{-n}, [\mathbf{w}_0, \mathbf{v}_{-1}]] = [\mathbf{w}_0, -\mathbf{u}_{-(n+1)}] = 2\mathbf{w}_{-(n+1)},$$

$$[\mathbf{v}_{-1}, \mathbf{v}_{-n}] = [\mathbf{v}_{-1}, \tfrac{1}{2}[\mathbf{v}_0, \mathbf{u}_{-n}]] = \tfrac{1}{2}[\mathbf{v}_0, 2\mathbf{v}_{-(n+1)}] = 0,$$

$$[\mathbf{u}_{-1}, \mathbf{u}_{-n}] = [\mathbf{u}_{-n}, [\mathbf{w}_0, \mathbf{v}_{-1}]] = [2\mathbf{w}_{-n}, \mathbf{v}_{-1}] + [\mathbf{w}_0, -2\mathbf{v}_{-(n+1)}] = 0,$$

$$[\mathbf{w}_{-1}, \mathbf{w}_{-n}] = [\mathbf{w}_{-n}, \tfrac{1}{2}[\mathbf{w}_0, \mathbf{u}_{-1}]] = \tfrac{1}{2}[\mathbf{w}_0, -2\mathbf{w}_{-(n+1)}] = 0,$$

$$[\mathbf{u}_{-1}, \mathbf{v}_{-n}] = [\mathbf{v}_{-n}, [\mathbf{w}_0, \mathbf{v}_{-1}]] = [\mathbf{u}_{-n}, \mathbf{v}_{-1}] = -2\mathbf{v}_{-(n+1)},$$

$$[\mathbf{w}_{-1}, \mathbf{u}_{-n}] = [\mathbf{u}_{-n}, \tfrac{1}{2}[\mathbf{w}_0, \mathbf{u}_{-1}]] = \tfrac{1}{2}[2\mathbf{w}_{-n}, \mathbf{u}_{-1}] = -2\mathbf{w}_{-(n+1)},$$

$$[\mathbf{w}_{-1}, \mathbf{v}_{-n}] = [\mathbf{v}_{-n}, \tfrac{1}{2}[\mathbf{w}_0, \mathbf{u}_{-1}]] = \tfrac{1}{2}[\mathbf{u}_{-n}, \mathbf{u}_{-1}] + \tfrac{1}{2}[\mathbf{w}_0, 2\mathbf{v}_{-(n+1)}] = -\mathbf{u}_{-(n+1)},$$

which all agree with the commutators given in Theorem 6.5.

Finally, a simple induction proof extends the range of validity of the commutator table to $m \in \{-r, \ldots, -1, 0, \ldots\}$ and $n \in \mathbb{Z}$ for arbitrarily large r. This calculation mimics that used earlier to prove that the commutators hold when $m \in \{0, 1, \ldots, r\}$ and is not duplicated here.

Appendix E

An example featuring $\mathfrak{sl}(3, \mathbb{R})$

Many group-motivated methods for studying integrable equations intimately involve finite-dimensional simple Lie algebras. The algebra $\mathfrak{sl}(2, \mathbb{R})$ (or $\mathfrak{sl}(2, \mathbb{C})$) occurs frequently, mainly because it is the simple Lie algebra with lowest dimension, so that the corresponding differential equations should be easier to study than those associated with larger algebras. The algebra $\mathfrak{sl}(2, \mathbb{R})$ has appeared frequently throughout this work — in particular, in the study of M-projections (Section 4.4) and auto-Bäcklund transformations (Section 5.5). Also, the loop algebra constructed and studied in Chapter 6 is based on $\mathfrak{sl}(2, \mathbb{R})$. However, it is important to appreciate that other algebras are compatible with all of the techniques introduced and studied here. This appendix examines one example where the Lie algebra $\mathfrak{sl}(3, \mathbb{R})$ plays a pivotal role.

E.1 Boussinesq equation

Weiss [94], in his study of the Boussinesq equation

$$0 = 3y_{tt} + \left(y_{xx} - 2y^2\right)_{xx}, \tag{E.1}$$

recovered the modified Boussinesq equation [29]

$$0 = v_t - w_{xx} - v_x w - v w_x, \quad 0 = 3w_t + v_{xx} - v v_x + 3w w_x, \tag{E.2}$$

using singularity manifold analysis. He identified a discrete symmetry group of equations (E.2) of order three, with the nontrivial group elements acting via

$$\begin{pmatrix} v \\ w \end{pmatrix} \mapsto \begin{pmatrix} \tilde{v} \\ \tilde{w} \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} & \mp\frac{3}{2} \\ \pm\frac{1}{2} & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} v \\ w \end{pmatrix}.$$

One can mimic the construction of Example 5.22, using these discrete symmetries of the modified Boussinesq equation to construct an auto-Bäcklund transformation for the Boussinesq equation.

Equations (E.1) and (E.2) and the relationships between them are now studied using the machinery developed earlier. Analysis begins by identifying some nonlocal partial symmetry generators of the modified Boussinesq equation. Two obvious potentials of equations (E.2) are defined by

$$\begin{aligned} p_x &= v, \\ p_t &= w_x + vw, \\ q_x &= -3w, \\ q_t &= v_x - \tfrac{1}{2}v^2 + \tfrac{3}{2}w^2. \end{aligned} \tag{E.3}$$

As well as the three classical symmetry generators,

$$\partial_x, \quad \partial_t, \quad x\partial_x + 2t\partial_t - v\partial_v - w\partial_w,$$

of equations (E.2), the Wahlquist-Estabrook prolongation of this system described by equations (E.3) admits three nonlocal partial symmetry generators. They are

$$\mathbf{u}_1 = e^p\partial_v, \quad \mathbf{u}_2 = e^{-(p+q)/2}\left(\partial_v - \partial_w\right), \quad \mathbf{u}_3 = e^{-(p-q)/2}\left(\partial_v + \partial_w\right).$$

By taking any two of these vector fields and applying the technique of Proposition 5.8, one can construct an M-projection from the modified Boussinesq equation onto a system equivalent to the Boussinesq equation.

The M-projection induced by $\{\mathbf{u}_2, \mathbf{u}_3\}$ is constructed in detail. Similar treatment yields the M-projections induced by the other two pairings of nonlocal partial symmetry generators. In order to obtain an augmented prolongation of the modified Boussinesq equation which admits genuine nonlocal symmetry generators equivalent to $\mathbf{u}_2$ and $\mathbf{u}_3$, it proves sufficient to introduce r, s and m defined by

$$r_x = e^{-(p+q)/2},$$

$$
\begin{aligned}
r_t &= -\tfrac{1}{2}(v+w)e^{-(p+q)/2}, \\
s_x &= e^{-(p-q)/2}, \\
s_t &= \tfrac{1}{2}(v-w)e^{-(p-q)/2}, \\
m_x &= e^{-(p+q)/2}s, \\
m_t &= -\tfrac{1}{2}(v+w)e^{-(p+q)/2}s - e^{-p}.
\end{aligned}
\tag{E.4}
$$

The nonlocal partial symmetry generators prolong to

$$
\begin{aligned}
\mathbf{u}_2 &= e^{-(p+q)/2}(\partial_v - \partial_w) + r(\partial_p + 3\partial_q) - r^2\partial_r + (rs-m)\partial_s - mr\partial_m, \\
\mathbf{u}_3 &= e^{-(p-q)/2}(\partial_v + \partial_w) + s(\partial_p - 3\partial_q) + m\partial_r - s^2\partial_s,
\end{aligned}
$$

and the internal symmetry generators, ∂_p and ∂_q, of the original prolongation prolong to

$$
\mathbf{v}_1 = \partial_p - \tfrac{1}{2}r\partial_r - \tfrac{1}{2}s\partial_s - m\partial_m, \quad \mathbf{v}_2 = \partial_q - \tfrac{1}{2}r\partial_r + \tfrac{1}{2}s\partial_s.
$$

Further internal symmetry generators are

$$
\mathbf{w}_1 = \partial_r, \quad \mathbf{w}_2 = \partial_s + r\partial_m, \quad \mathbf{w}_3 = \partial_m.
$$

Finally, another nonlocal symmetry generator can be found. It is

$$
\begin{aligned}
\mathbf{u}_4 &= [\mathbf{u}_2, \mathbf{u}_3] \\
&= e^{-(p-q)/2}r(\partial_v + \partial_w) - e^{-(p+q)/2}s(\partial_v - \partial_w) \\
&\quad + (rs-2m)\partial_p - 3rs\partial_q + mr\partial_r + s(m-rs)\partial_s + m^2\partial_m.
\end{aligned}
$$

The symmetry algebra spanned by $\{\mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4, \mathbf{v}_1, \mathbf{v}_2, \mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$ leads to an HC-projection of the system of equations comprising equations (E.2) to (E.4). Solutions to the corresponding HC-projected problem can be described by

$$
\begin{aligned}
v_x &= -\tfrac{1}{4}v^2 - \tfrac{3}{4}w^2 + y(x,t), \\
w_{xx} &= -\tfrac{3}{2}vw_x - \tfrac{3}{8}v^2w + \tfrac{7}{8}w^3 - \tfrac{1}{2}wy(x,t) + z(x,t),
\end{aligned}
\tag{E.5}
$$

where the unknown functions y and z are determined by

$$
0 = y_t - z_x, \quad 0 = 3z_t + y_{xxx} - 4yy_x.
$$

Eliminating z between these equations recovers equation (E.1). Notice that equations (E.5) correspond to the Miura transformation given by Weiss (equations (2.21) in [94]) and found originally by Fokas and Anderson [29].

Let $\mathfrak{g}$ be the Lie algebra spanned by $\{\mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4, \mathbf{v}_1, \mathbf{v}_2, \mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$. The commutator table for $\mathfrak{g}$ is

	$\mathbf{u}_2$	$\mathbf{u}_3$	$\mathbf{u}_4$	$\mathbf{v}_1$	$\mathbf{v}_2$	$\mathbf{w}_1$	$\mathbf{w}_2$	$\mathbf{w}_3$
$\mathbf{u}_2$	0	$\mathbf{u}_4$	0	$\frac{1}{2}\mathbf{u}_2$	$\frac{1}{2}\mathbf{u}_2$	$-\mathbf{v}_1 - 3\mathbf{v}_2$	0	$\mathbf{w}_2$
$\mathbf{u}_3$		0	0	$\frac{1}{2}\mathbf{u}_3$	$-\frac{1}{2}\mathbf{u}_3$	0	$-\mathbf{v}_1 + 3\mathbf{v}_2$	$-\mathbf{w}_1$
$\mathbf{u}_4$			0	$\mathbf{u}_4$	0	$-\mathbf{u}_3$	$\mathbf{u}_2$	$2\mathbf{v}_1$
$\mathbf{v}_1$				0	0	$\frac{1}{2}\mathbf{w}_1$	$\frac{1}{2}\mathbf{w}_2$	$\mathbf{w}_3$
$\mathbf{v}_2$					0	$\frac{1}{2}\mathbf{w}_1$	$-\frac{1}{2}\mathbf{w}_2$	0
$\mathbf{w}_1$						0	$\mathbf{w}_3$	0
$\mathbf{w}_2$							0	0
$\mathbf{w}_3$								0

so that $\mathfrak{g} \cong \mathfrak{sl}(3,\mathbb{R})$. Furthermore, $\mathfrak{g}$ splits into the vector space direct sum

$$\mathfrak{g} = \mathfrak{u} + \mathfrak{v} + \mathfrak{w},$$

where $\mathfrak{u}$, $\mathfrak{v}$ and $\mathfrak{w}$ are the *subalgebras* $\mathrm{sp}\{\mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4\}$, $\mathrm{sp}\{\mathbf{v}_1, \mathbf{v}_2\}$ and $\mathrm{sp}\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$ respectively. With respect to the standard matrix representation of $\mathfrak{sl}(3,\mathbb{R})$ as 3×3 matrices with zero trace, this corresponds to the factorization into subalgebras of strictly lower triangular, diagonal and strictly upper triangular matrices for $\mathfrak{u}$, $\mathfrak{v}$ and $\mathfrak{w}$ respectively. For later use, notice that $\mathfrak{u}$ comprises the nonlocal symmetry generators of the prolongation, while $\mathfrak{v}$ contains those internal symmetry generators inherited from internal symmetry generators of the original (p,q)-prolongation. Finally, $\mathfrak{w}$ is made up of those internal symmetry generators involving only the pseudopotentials introduced when augmenting the original prolongation.

The transformation found by Fokas and Anderson can thus be recovered as an M-projection of the modified Boussinesq equation involving a symmetry algebra isomorphic to $\mathfrak{sl}(3,\mathbb{R})$. This should be compared to the Miura transformation relating the KdV and modified KdV equations. In that situation, the construction of the M-projection of the mKdV equation features a symmetry algebra isomorphic to $\mathfrak{sl}(2,\mathbb{R})$. The similarities between the two pairs of equations do not rest there, however. Further analysis sheds some light on the role Lie algebras can play when studying integrable equations.

E.2 The case of $\mathfrak{sl}(2,\mathbb{R})$ reviewed

Examples 5.6, 5.9 and 5.3 have shown that the prolongation of the mKdV equation

$$0 = v_t + v_{xxx} - 6v^2 v_x \tag{E.6}$$

described by

$$w_x = v, \quad w_t = -v_{xx} + 2v^3, \tag{E.7}$$

admits two nonlocal partial symmetry generators. They are $e^{2w}\partial_v$ and $e^{-2w}\partial_v$, and either one yields an M-projection onto the KdV equation. For instance, after augmenting the prolongation by introducing y defined by

$$y_x = e^{2w}, \quad y_t = 2e^{2w}(-v_x + v^2), \tag{E.8}$$

one finds the genuine nonlocal symmetry generator

$$\mathbf{u}_1 = e^{2w}\partial_v + y\partial_w + y^2\partial_y$$

equivalent to $e^{2w}\partial_v$. Furthermore, the internal symmetry generator ∂_w of the original prolongation prolongs to $\mathbf{v}_1 = \partial_w + 2y\partial_y$, while another internal symmetry generator is $\mathbf{w}_1 = \partial_y$. Letting $\mathfrak{g}$ denote the symmetry algebra with basis $\{\mathbf{u}_1, \mathbf{v}_1, \mathbf{w}_1\}$, the G-induced HC-projection of the system of equations (E.6) to (E.8) is described by

$$v_x = v^2 + 2u(x,t),$$

where the unknown function u must satisfy the KdV equation

$$0 = u_t + u_{xxx} + 12uu_x.$$

Notice that $\mathfrak{g} \cong \mathfrak{sl}(2,\mathbb{R})$ can be expressed as a vector space direct sum

$$\mathfrak{g} = \mathfrak{u} + \mathfrak{v} + \mathfrak{w},$$

where $\mathfrak{u}$, $\mathfrak{v}$ and $\mathfrak{w}$ are the *subalgebras* $\mathrm{sp}\{\mathbf{u}_1\}$, $\mathrm{sp}\{\mathbf{v}_1\}$ and $\mathrm{sp}\{\mathbf{w}_1\}$ respectively. With respect to the standard matrix representation of $\mathfrak{sl}(2,\mathbb{R})$ as 2×2 matrices with zero trace, this corresponds to the factorization into subalgebras of strictly lower triangular, diagonal and strictly upper triangular matrices for $\mathfrak{u}$, $\mathfrak{v}$ and $\mathfrak{w}$ respectively.

Notice also that $\mathfrak{u}$ comprises the nonlocal symmetry generator of the prolongation, while $\mathfrak{v}$ contains the internal symmetry generator inherited from the internal symmetry generator of the original w-prolongation. Finally, $\mathfrak{w}$ is made up of the internal symmetry generator involving only the pseudopotential introduced when augmenting the original prolongation.

This situation is remarkably similar to that described in the previous section. From a symmetry group-theoretical point of view the M-projections onto the Boussinesq equation and the KdV equation are obviously closely related. The symmetry algebras involved admit comparable factorizations into vector space direct sums of subalgebras. The only difference is that $\mathfrak{sl}(2,\mathbb{R})$ is involved in the projection onto the KdV equation, while $\mathfrak{sl}(3,\mathbb{R})$ appears in the projection onto the Boussinesq equation.

E.3 Singularity manifold equations

A key component of an M-projection is the differential equation which admits both the source and target equations of the M-projection as HC-projections. This differential equation arises as the Wahlquist-Estabrook prolongation of the source equation which appears when constructing the M-projection. In the case of the Miura transformation, this system is the prolongation of the mKdV equation described by equations (E.6) to (E.8). As shown in Example 5.3, this system is equivalent to the PPmKdV equation

$$0 = y_t + y_{xxx} - \tfrac{3}{2} y_x^{-1} y_{xx}^2, \tag{E.9}$$

together with several of its differential consequences. This equation is actually the one obtained from the KdV equation via Painlevé analysis and admits a subalgebra of symmetries isomorphic to $\mathfrak{sl}(2,\mathbb{R})$ [93]. It is sometimes called the singularity manifold equation for the KdV equation. The system arising from the M-projection onto the Boussinesq equation comprises equations (E.2) to (E.4). Following the substitutions $v = p_x$ and $w = -\frac{1}{3} q_x$, equations (E.2) and (E.3) reduce to

$$0 = 3p_t + q_{xx} + p_x q_x, \quad 0 = q_t - p_{xx} + \tfrac{1}{2} p_x^2 - \tfrac{1}{6} q_x^2,$$

and some differential consequences of these equations. Equations (E.4) then define a Wahlquist-Estabrook prolongation of this system, which is itself simplified by the substitutions $p = -\log(r_x s_x)$ and $q = \log(s_x/r_x)$. These yield the system of equations

$$
\begin{aligned}
0 &= 6r_t - \left(\frac{3r_x^2 - s_x^2}{r_x^2}\right) r_{xx} - \left(\frac{3r_x^2 + s_x^2}{r_x s_x}\right) s_{xx}, \\
0 &= 6s_t + \left(\frac{s_x(3r_x^2 + s_x^2)}{r_x^3}\right) r_{xx} + \left(\frac{3r_x^2 - s_x^2}{r_x s_x}\right) s_{xx},
\end{aligned}
\tag{E.10}
$$

together with

$$m_x = r_x s, \quad m_t = r_t s - r_x s_x,$$

which define m as a pseudopotential of equations (E.10). Eliminating s via $s = m_x/r_x$ leads to the system of equations

$$
\begin{aligned}
0 &= m_t + m_{xx} - \frac{2m_x(m_{xxx}r_x - m_x r_{xxx})}{3(m_{xx}r_x - m_x r_{xx})}, \\
0 &= r_t + r_{xx} - \frac{2r_x(m_{xxx}r_x - m_x r_{xxx})}{3(m_{xx}r_x - m_x r_{xx})}.
\end{aligned}
\tag{E.11}
$$

Recall that the PPmKdV equation features a symmetry algebra with a subalgebra isomorphic to $\mathfrak{sl}(2, \mathbb{R})$ and admits HC-projections onto the KdV and mKdV equations. Equations (E.11) feature a symmetry algebra isomorphic to $\mathfrak{sl}(3, \mathbb{R})$ and admit HC-projections onto the Boussinesq and modified Boussinesq equations. For later use, these symmetry generators are listed below:

$$\mathbf{u}_2 = -mr\partial_m - r^2\partial_r, \quad \mathbf{u}_3 = m\partial_r, \quad \mathbf{u}_4 = m^2\partial_m + mr\partial_r,$$

$$\mathbf{v}_1 = -m\partial_m - \tfrac{1}{2}r\partial_r, \quad \mathbf{v}_2 = -\tfrac{1}{2}r\partial_r,$$

$$\mathbf{w}_1 = \partial_r, \quad \mathbf{w}_2 = r\partial_m, \quad \mathbf{w}_3 = \partial_m.$$

The labelling is consistent with the internal and nonlocal symmetry generators of the Wahlquist-Estabrook prolongation of the modified Boussinesq equation, introduced in Section E.1, which is equivalent to equations (E.11).

In terms of M-projections it has been shown that equations (E.11) and the PPmKdV equation play analogous roles. However, this relationship does not carry over to Painlevé analysis. While the PPmKdV equation is the singularity manifold equation associated with the KdV equation, the singularity manifold equation for the Boussinesq equation is actually

$$0 = 3\left(\frac{\phi_t}{\phi_x}\right)_t + \left(\{\phi; x\} + \frac{3}{2}\left(\frac{\phi_t}{\phi_x}\right)^2\right)_x$$

where

$$\{\phi; x\} = \left(\frac{\phi_{xx}}{\phi_x}\right)_x - \frac{1}{2}\left(\frac{\phi_{xx}}{\phi_x}\right)^2$$

is the Schwarzian derivative [94]. This equation does arise as an HC-projection of equations (E.11), however. The internal symmetry generators $\mathbf{w}_2$ and $\mathbf{w}_3$ of equations (E.11) induce an HC-projection described by

$$m_{xx} = m_x r_x^{-1} r_{xx} + r_x^{1/2}\exp(\alpha/2), \quad m_t = \tfrac{1}{3} m_x \alpha_x - r_x^{1/2}\exp(\alpha/2),$$

where $\alpha(x,t)$ must satisfy an HC-projected system of the form

$$0 = 3r_t - r_x\alpha_x, \quad 0 = 6r_x^2\alpha_t + r_x^2\alpha_x^2 + 6r_x r_{xxx} - 9r_{xx}^2. \tag{E.12}$$

This system possesses a symmetry generator ∂_α which is the projection of $\frac{1}{2}(\mathbf{v}_2 - \mathbf{v}_1)$. The corresponding HC-projected problem can be described by

$$0 = 3\left(\frac{r_t}{r_x}\right)_t + \left(\{r; x\} + \frac{3}{2}\left(\frac{r_t}{r_x}\right)^2\right)_x, \tag{E.13}$$

which is obtained by eliminating α between equations (E.12). Consequently, $r(x,t)$ satisfies the singularity manifold equation which can be interpreted as the HC-projection of equations (E.11) induced by the symmetry group with infinitesimal generators $\{\mathbf{v}_2 - \mathbf{v}_1, \mathbf{w}_2, \mathbf{w}_3\}$. By appealing to the obvious symmetry $(m, r) \mapsto (r, m)$ in equations (E.11), one finds that the symmetry generators $\{\mathbf{u}_3, \mathbf{v}_2, \mathbf{w}_1\}$ lead to an HC-projection from equations (E.11) onto

$$0 = 3\left(\frac{m_t}{m_x}\right)_t + \left(\{m; x\} + \frac{3}{2}\left(\frac{m_t}{m_x}\right)^2\right)_x. \tag{E.14}$$

Thus, $m(x,t)$ must also satisfy the singularity manifold equation. It follows that if $\{m(x,t), r(x,t)\}$ is a solution to equations (E.11) then $m(x,t)$ and $r(x,t)$ must both satisfy the singularity manifold equation, respectively equations (E.14) and (E.13). That is, included among the differential consequences of equations (E.11) are two copies of the singularity manifold equation!

A further difference between the singularity manifold equations is their relationships with the KdV and Boussinesq equations. The KdV equation arises as an HC-projection of the PPmKdV equation induced by a symmetry group with algebra isomorphic to $\mathfrak{sl}(2,\mathbb{R})$. However the same cannot be said for the Boussinesq equation. Because the Boussinesq equation arises as an eighth order HC-projection of equations (E.11), and since equation (E.13) is a third order HC-projection of the same system it follows that the Boussinesq equation admits a Wahlquist-Estabrook prolongation equivalent to equation (E.13). This prolongation features a five-dimensional pseudopotential space. After constructing this prolongation using the methods of Section 4.4, one finds that the internal symmetry algebra is three-dimensional. Thus, the prolongation cannot possibly admit a full internal symmetry group and Theorem 4.4 implies that the Boussinesq equation is not an HC-projection of the singularity manifold equation. The HC-projections of equations (E.11) which have been discussed above are displayed in Figure E.1. Only the dimension of the appropriate symmetry group is used to label each HC- or M-projection, which is denoted by a solid or broken line respectively.

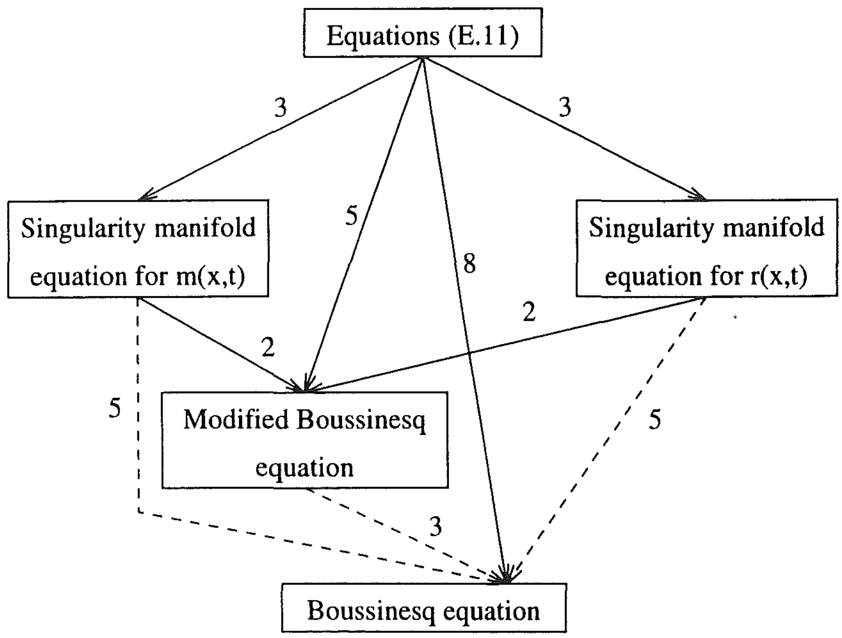

Figure E.1: Lattice diagram of equations related to the Boussinesq equation by HC- and M-projections

The research included in this appendix is admittedly incomplete. It raises some interesting questions about the behaviour of M-projections and Painlevé analysis as the underlying symmetry groups become larger, as well as providing further evidence of the utility of HC-projections as an interpretive tool. Investigations are continuing.

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